ELSEVIER Journal of Geometry and Physics 36 (2000) 285–314

JOURNAL OF GEOMETRY AND PHYSICS

Geodesic connectedness and conjugate points in GRW space–times

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Received 2 March 2000

Abstract

Given two points of a generalized Robertson–Walker space–time, the existence, multiplicity and causal character of geodesics connecting them is characterized. Conjugate points of such geodesics are related to conjugate points of geodesics on the fiber, and Morse-type relations are obtained. Applications to bidimensional space–times and to GRW space–times satisfying the timelike convergence condition are also found. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 83C15; 53C22; 53C50
Subj. Class.: Differential geometry, General relativity
Keywords: Geodesic connectedness; Conjugate points; Space–times

1. Introduction

Recently, geodesic connectedness of Lorentzian manifolds has been widely studied, and some related questions appear which invoke great interest; among them are (i) how to determine the existence, multiplicity and causal character of geodesics connecting two points, and (ii) how to study their conjugate points and to find Morse-type relations. These questions have been answered, totally or partially, for stationary or splitting manifolds (see, e.g., [4,11,13,15,21]). Our purpose is to answer them totally in the class of generalized Robertson–Walker (GRW) space–times.

GRW space–times (see Section 2 for precise definitions) are warped products $(I \times F, g^f = -\mathrm{d}t^2 + f^2 g)$ which generalize Robertson–Walker ones because no assumption on their fiber is done, and they have interesting properties from both the mathematical

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PII: S0393-0440(00)00027-9

and the physical point of view [1,17–19]. GRW space–times are also particular cases of *multiwarped space–times*, whose geodesic connectedness has been recently studied by using a topological method [8]. They can be also seen as splitting type manifolds, studied in [15, Chapter 8], or as a type of Reissner–Nordström intermediate space–times, studied in [9,10]. Nevertheless, we will see here that the results for GRW space–times can be obtained in a simpler approach, and are sharper. In fact, we will develop the following direct point of view.

Given a geodesic $\gamma(t) = (\tau(t), \gamma_F(t))$ of the GRW space–time, the component γ_F is a pregeodesic of its fiber. So, if $\mathrm{d}\tau/\mathrm{d}t$ does not vanish, we can consider the reparameterization $\gamma_F(\tau)$, and γ will cross a point $z_0 = (\tau_0, x_0)$ if and only if $x_0 = \gamma_F(\tau_0)$. This simple fact yields a result on connectedness by timelike and causal geodesics [17, Theorems 3.3 and 3.7]. For spacelike geodesics, the reparameterization $\gamma_F(\tau)$ may fail. This problem can be skipped sometimes by simple arguments on continuity [18, Theorem 3.2], but we will study it systematically in order to solve completely the problem of geodesic connectedness. Moreover, this will also be the key to solve the other related problems (multiplicity, conjugate points, etc.).

After some preliminaries in Section 2, we state the conditions for geodesic connectedness in Section 3. In fact, we give three Conditions (A), (B), (C) of increasing generality, and a fourth Condition (R) which covers a residual case. All these conditions are imposed on the warping function f; on the fiber, we assume just a weak condition on convexity (each two points x_0, x_0' can be joined by a minimizing F-geodesic $\hat{\gamma}_F$), which is known to be completely natural (see [17, Remark 3.2]). These conditions are somewhat cumbersome, because they yield not only sufficient but also necessary hypotheses for geodesic connectedness; however, they yield very simple sufficient conditions. For example (Lemmas 3 and 9), *if the GRW space–time is not geodesically connected then f must admit a limit at some extreme of the interval* $I = (a, b)$; *if this extreme is* b (*resp.* a) *then* f' *must be strictly positive* (*resp. negative*) *in a non-empty subinterval* $(\bar{b}, b) \subseteq (a, b)$ (*resp.* $(a, \bar{a}) \subseteq (a, b)$) (moreover, in this case Table 1 can be used). Condition (A) summarizes in which cases the warping function f has a "good behavior" at the extremes of $I = (a, b)$ in order to obtain

Table 1
If f is continuously extendible to b, when Condition (A) is satisfied at b

	$b < \infty$			$b = \infty$	
	$\lim_{\tau \to b} f(\tau)$		Condition (A)	$\lim_{\tau \to b} f(\tau)$	Condition (A)
1		0	Yes	0	Yes
2	$\alpha \in \mathbb{R},\ \alpha \neq 0$	f' not extendible to b	No information	$\alpha \in \mathbb{R},\ \alpha \neq 0$	Yes
		$\lim_{\tau \to b} f' = \beta \in [-\infty, 0)$	Yes		
		$\lim_{\tau \to b} f' = \beta \in (0, \infty]$	No[a]		
		$\lim_{\tau \to b} f' = 0$ and f'' bounded in $[b - \epsilon, b)$	Yes		
3		∞	No	$\infty,\ \int_c^b \frac{1}{f} = \infty$	Yes
				$\infty,\ \int_c^b \frac{1}{f} < \infty$	No[b]

[a] Condition (C) does not hold either. No information on Condition (R), if applicable.
[b] No information on Condition (C) or (R).

geodesic connectedness. This condition is equal to the one obtained in [8] for multiwarped space–times; nevertheless, we will reprove it because a simpler proof is now available and the ideas in this proof will be used in the following more general conditions. Condition (B) takes into account that when the diameter of the fiber is finite, even a "not so good" behavior of f at an extreme, say b, may allow the following situation: a fixed point $z_0 = (\tau_0, x_0)$ can be connected to $z'_0 = (\tau'_0, x'_0)$, where τ'_0 is close enough to b, by means of a geodesic $\gamma(t) = (\tau(t), \gamma_F(t))$ such that $\tau(t)$ points out from τ_0 to b and, perhaps "bounces" close to b. Condition (C) takes into account that even when Condition (B) does not hold, the following situation in the previous case may hold: a geodesic which points out from τ_0 to the extreme a, bounces close to a and comes back towards b, may connect z_0 and z'_0. Condition (C) is shown to be the more general condition for geodesic connectedness, except in the case: if the limit of f at both extremes a, b is equal to the supremum of f, and this supremum is not reached at I, then z_0 and z'_0 perhaps could be joined by geodesics which bounce many times close to a and b. Examples of the strict implications between the different conditions are provided. In Section 3, we also state our results on existence of connecting geodesics, which are proven in Section 4:

1. Either Condition (C) or Condition (R) is sufficient for geodesic connectedness (Theorem 1).
2. If we assume a stronger condition of convexity on the fiber (each geodesic $\hat{\gamma}_F$ above is assumed to be the only geodesic which connects x_0, x'_0), then one of the two Conditions (C) or (R) is also necessary (Theorem 2). The necessity of this stronger convexity assumption is also discussed.
3. Under Condition (A) (or, even in some cases (B)), if the topology of F is not trivial then each two points $z_0 = (\tau_0, x_0),\ z'_0 = (\tau'_0, x'_0)$ can be joined by infinitely many spacelike geodesics (Theorem 3).
4. For causal geodesics: (i) if z_0 and z'_0 are causally related then there exist a causal geodesic joining them (this result was previously proven in [17]), (ii) if z_0 and z'_0 are not conjugate (or even if just x_0 and x'_0 are not conjugate which will be shown to be less restrictive), then there are at most finitely many timelike geodesics joining them (Theorem 3), and (iii) if the fiber is strongly convex, then there exist at most one connecting causal geodesic (Theorem 2).

This machinery is used in Section 5 to obtain a precise relation between the conjugate points of a geodesic in $I \times F$ and its projection on F (Theorem 4, Corollary 1). From this result, Morse-type relations which relate the topology of the space of curves joining two non-conjugate points and the Morse indexes of the geodesics joining them are obtained (see Corollary 2 and the discussion above it). We remark that, Morse indexes are defined here in the geometrical sense "sum of the orders of conjugate points" because, for any spacelike geodesic, its index form is positive definite and negative definite on infinite-dimensional subspaces (if $\dim F > 1$). About this kind of problem, the following previous references should be taken into account. Conjugate points of null geodesics in globally hyperbolic space–times were studied by Uhlenbeck [21], and we also make some remarks in Section 5 relating our results. In a general setting, conjugate points on spacelike geodesics were studied by Helfer [14], who also considered the Maslov index of a geodesic. He showed that these conjugate points may have very different properties to conjugate points for Riemannian manifolds (instability, non-isolation, etc.), but these problems can be skipped in our study. In [6] (see also [15, Section 5]), an attempt to obtain a Morse theory for standard stationary manifolds is carried out, and in [13], an index theorem (in terms of the Maslov index) applicable in particular to stationary manifolds is obtained. On the other hand, some recent papers studied Morse theory for timelike or lightlike geodesics joining a point and a timelike curve (see [12] and references therein). Typically, these results are stated for strongly causal space–times (including all GRW space–times), and they need an assumption on *coercivity* which does not necessarily hold under our hypotheses. It is not difficult to check that our results are also applicable to face this problem.

In Section 6, we particularize the previous results to two cases. First, Section 6.1, when the fiber is also an interval of $\mathbb{R}$. In this bidimensional case, the opposite metric $-g^f$ is standard static, and we reobtain and extend the theorem in [5]. We recall that the proof in this reference is obtained by a completely different method, which relies on the function spectral flow on a geodesic (see Remark (2) in Theorem 4 for noteworthy comments about this approach). Finally, in Section 6.2, we consider the case $\mathrm{Ric}(\partial_t, \partial_t) \geq 0$. This condition is natural from a physical point of view. In fact, the stronger condition $\mathrm{Ric}(v, v) \geq 0$ for all timelike v is called the timelike convergence condition, and says that gravity, on an average, attracts. Condition $\mathrm{Ric}(\partial_t, \partial_t) \geq 0$ is equivalent to $f'' \leq 0$, and this inequality implies Condition (A) if f cannot be continuously extended to positive values at any extreme. Corollary 6 summarizes our results in this case. We finish with an extension, in our ambient, of a result in [21, Corollary 7].

2. Preliminaries

Let (F, g) be a Riemannian manifold, $(I, -\mathrm{d}\tau^2)$ an open interval of $\mathbb{R}$ with $I = (a, b)$ and its usual metric reversed, and $f > 0$ a smooth function on I. A GRW space–time with base $(I, -\mathrm{d}\tau^2)$, fiber (F, g) and warping function $f > 0$ is the product manifold $I \times F$ endowed with the Lorentz metric

$$g^f = -\pi_I^* \,\mathrm{d}\tau^2 + (f \circ \pi_I)^2 \pi_F^* g \equiv -\mathrm{d}\tau^2 + f^2 g, \tag{1}$$

where π_I and π_F are the natural projections of $I \times F$ onto I and F, respectively, and will be omitted when there is no possibility of confusion.

A Riemannian manifold will be called weakly convex if any two of its points can be joined by a geodesic which minimize the distance; if, in addition this geodesic is the only one which joins the two points it will be called strongly convex (recall that these names do not coincide with those in [17]). Of course, if the Riemannian manifold (F, g) is complete then it is weakly convex by the Hopf–Rinow theorem, but the converse is not true (a detailed study of when a (incomplete) Riemannian manifold is weakly convex can be seen in [2]). It is well known that Cartan–Hadamard manifolds (i.e. complete, simply connected and with non-positive curvature) are strongly convex and, of course, so are locally all Riemannian manifolds (more results on strong convexity can be seen in [11]). We will denote by d

the distance on F canonically associated to the Riemannian metric g, and by diam(F) its diameter (the supremum, possibly infinity, of the d-distances between points of F).

Given a vector X tangent to $I \times F$ we will say that X is timelike (resp. lightlike, causal, spacelike) if $g^f(X, X) < 0$ (resp. $= 0$, ≤ 0, > 0); the timelike vector field $\partial/\partial\tau$ fixes the canonical future orientation in $I \times F$. Given $z, z' \in I \times F$, we will say that they are causally (resp. chronologically) related if they can be joined (z with z' or vice versa) by a future-pointing non-spacelike (resp. timelike) piecewise smooth curve.

Let $\gamma : \mathcal{J} \to I \times F$, $\gamma(t) = (\tau(t), \gamma_F(t))$ be a (smooth) curve on the interval $\mathcal{J}$. It is well known that γ is a geodesic with respect to g^f if and only if

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}t^2} = -\frac{c}{f^3 \circ \tau}\frac{\mathrm{d}f}{\mathrm{d}\tau} \circ \tau, \tag{2}$$

$$\frac{D}{\mathrm{d}t}\frac{\mathrm{d}\gamma_F}{\mathrm{d}t} = -\frac{2}{f \circ \tau}\frac{\mathrm{d}(f \circ \tau)}{\mathrm{d}t}\frac{\mathrm{d}\gamma_F}{\mathrm{d}t} \tag{3}$$

on $\mathcal{J}$, where $D/\mathrm{d}t$ denotes the covariant derivative associated to γ_F and c the constant $(f^4 \circ \tau)g(\mathrm{d}\gamma_F/\mathrm{d}t,\ \mathrm{d}\gamma_F/\mathrm{d}t)$. From (2),

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \epsilon\left(-D + \frac{c}{f^2 \circ \tau}\right)^{1/2} \tag{4}$$

with $D = g^f(\mathrm{d}\gamma/\mathrm{d}t,\ \mathrm{d}\gamma/\mathrm{d}t)$ and $\epsilon \in \{\pm 1\}$. Note that, if $c = 0$ then $\mathrm{d}^2\tau/\mathrm{d}t^2 \equiv 0$, i.e. the geodesics on the base I are naturally lifted to geodesics of the GRW space–time as in any warped product. For all the other geodesics, it is natural to normalize choosing them with $c = 1$. This normalization will be always chosen except in Section 5 where the formulae will be explicitly taken with a different normalization. All geodesics will also be assumed inextendible, i.e. with a maximal domain.

By Eq. (3), each (non-constant) γ_F is a pregeodesic of (F, g), so if we consider the reparameterization $\hat{\gamma}_F(r) = \gamma_F(t(r))$, where

$$\frac{\mathrm{d}t}{\mathrm{d}r} = f^2 \circ \tau \circ t \tag{5}$$

(in a maximal domain) we obtain that $\hat{\gamma}_F$ is a geodesic of (F, g) being

$$g\left(\frac{\mathrm{d}\hat{\gamma}_F}{\mathrm{d}r}, \frac{\mathrm{d}\hat{\gamma}_F}{\mathrm{d}r}\right) = 1. \tag{6}$$

From now on, we will assume that (F, g) is weakly convex for any result where geodesic connectedness is involved; such assumption has proven to be completely natural [8,17,18]. In fact, as an immediate consequence of (2) and (3) we get the following lemma.

Lemma 1. *There exists a geodesic joining* $z_0 = (\tau_0, x_0)$ *and* $z'_0 = (\tau'_0, x'_0)$, $\tau_0 \leq \tau'_0$ *if* $\tau_0 = \tau'_0$ *and* $(\mathrm{d}/\mathrm{d}\tau)1/f^2(\tau_0) = 0$.

Now, the case when the geodesic $\hat{\gamma}_F$ can be reparameterized by using $\tau \in (a_*, b_*)$ as a parameter (for some interval (a_*, b_*)) will be considered. Putting $\tilde{\gamma}(\tau) \equiv \hat{\gamma} \circ r(\tau)$ we have

$\mathrm{d}\tilde{\gamma}/\mathrm{d}\tau = h^{\epsilon}_{D}(\tau)\mathrm{d}\hat{\gamma}/\mathrm{d}r$ where $h^{\epsilon}_{D} \equiv h^{\epsilon} : (a_*, b_*) \subseteq I \to \mathbb{R}$ is defined as

$$h^{\epsilon} = \epsilon f^{-2}\left(-D + \frac{1}{f^2}\right)^{-1/2}, \tag{7}$$

and $\epsilon = \pm 1$. When this reparameterization can be done in such a way that if τ goes from τ_0 to τ_0', the integral of h_ϵ is exactly equal to the distance between $x_0, x_0' \in F$, then a geodesic joining (τ_0, x_0) and (τ_0', x_0') can be constructed, yielding the following lemma.

Lemma 2. *There exists a geodesic connecting* $z_0 = (\tau_0, x_0)$ *and* $z_0' = (\tau_0', x_0')$, $\tau_0 \leq \tau_0'$ *if there is a constant* $D \in \mathbb{R}$ $(D = g^f(\mathrm{d}\gamma/\mathrm{d}t,\ \mathrm{d}\gamma/\mathrm{d}t))$ *such that*

1. *either* $1/f^2(\tau_0) \neq D$ *or if this equality holds then* $(\mathrm{d}/\mathrm{d}\tau)1/f^2(\tau_0) \neq 0$, *and*
2. *the maximal domain* (a_*, b_*) *of* h_ϵ *includes* (τ_0, τ_0'), *and*

$$\int_{\tau_0}^{\tau_0'} h^{\epsilon} = L, \tag{8}$$

where $L = d(x_0, x_0')$.

(In this case $\epsilon = 1$.) When the reparameterization $\tilde{\gamma}(\tau)$ fails then the points where the denominator of h^{ϵ} goes to zero must be specially taken into account. Firstly, we will specify the maximal domain of h^{ϵ}. Fix $D \in \mathbb{R}$ such that $1/f^2(\tau_0) \geq D$ and consider the subsets

$$A_+ = \left\{\tau \in (a,b) : \tau_0 \leq \tau,\ \frac{1}{f^2(\tau)} = D\right\} \cup \{b\}, \tag{9}$$

$$A_- = \left\{\tau \in (a,b) : \tau_0 \geq \tau,\ \frac{1}{f^2(\tau)} = D\right\} \cup \{a\}. \tag{10}$$

Define $a_* \equiv a_*(D)$, $b_* \equiv b_*(D)$ by

$$\begin{aligned} &\text{If}\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{f^2}(\tau_0) > 0, \quad \text{then} \quad b_* = \min(A_+ - \{\tau_0\}), \quad a_* = \max(A_-),\\ &\text{If}\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{f^2}(\tau_0) < 0, \quad \text{then} \quad b_* = \min(A_+), \quad a_* = \max(A_- - \{\tau_0\}),\\ &\text{If}\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{f^2}(\tau_0) = 0, \quad \text{then} \quad b_* = \min(A_+), \quad a_* = \max(A_-). \end{aligned} \tag{11}$$

Now, it is not difficult to check that Lemma 2 also holds if we assume the following convention for the integral (8).

Convention 1. From now on integral (8) will be understood in the following generalized sense: for $\epsilon = 1$, if $\int_{\tau_0}^{b_*} h^{\epsilon=1} \geq L$, then the first member of (8) denotes the usual integral and we will also follow the notation ${}_{+[0]}\int_{\tau_0}^{\tau_0'} h^{\epsilon}$; otherwise and if $b_* \neq b$, we can follow integrating by reversing the sense of integration (recall $\tau_0' \leq b_*$) and, if $\int_{\tau_0}^{b_*} h^{\epsilon=1} - \int_{b_*}^{a_*} h^{\epsilon=1} \geq L$, then the first member of (8) means $\int_{\tau_0}^{b_*} h^{\epsilon=1} - \int_{b_*}^{\tau_0'} h^{\epsilon=1}$ which we denote by ${}_{+[1]}\int_{\tau_0}^{\tau_0'} h^{\epsilon}$. If this last inequality does not hold and $a_* \neq a$, then the procedure must follow reversing the sense of integration ($\tau_0' \geq a_*$) as many times as necessary in the obvious way. Analogously, when $\epsilon = -1$, first member of (8) means either $\int_{\tau_0}^{\tau_0'} h^{\epsilon=-1} \equiv_{-[0]} \int_{\tau_0}^{\tau_0'} h^{\epsilon}$ (in this case if $\tau_0 < \tau_0'$ the integral is negative; so equality (8) cannot hold) or $\int_{\tau_0}^{a_*} h^{\epsilon=-1} - \int_{a_*}^{\tau_0'} h^{\epsilon=-1} \equiv_{-[1]} \int_{\tau_0}^{\tau_0'} h^{\epsilon}$ or $\int_{\tau_0}^{a_*} h^{\epsilon=-1} - \int_{a_*}^{b_*} h^{\epsilon=-1} + \int_{b_*}^{\tau_0'} h^{\epsilon=-1} \equiv_{-[2]} \int_{\tau_0}^{\tau_0'} h^{\epsilon}$, etc.

Remark 1. *From* (7), *fixed* $\epsilon \in \{\pm 1\}$, *for each* $D \in \mathbb{R}$ *we have at most one* τ_0' *such that Eq.* (8) *holds, possibly under Convention* 1. *Let us introduce the parameter* $K \equiv K(D, \epsilon)$ *by means of* $K = 1/f^2(\tau_0) - D$ *if* $\epsilon = 1$, $K = D - 1/f^2(\tau_0)$ *if* $\epsilon = -1$. *So, for fixed L, a function* $\tau(K) = \tau_0'$ *is defined for K in a certain domain* $\mathcal{D}$ *of* $\mathbb{R}$.

3. Conditions for geodesic connectedness

Now, we are ready to establish four conditions (Conditions (A), (B), (C), and (R)) on the warping function f which, independently, ensure the geodesic connectedness of the GRW space–time (Lemmas 8 and 9 and Theorem 1). Roughly, Condition (A) implies not only the geodesic connectedness but also that every $(\tau_0, x_0) \in I \times F$ can be joined with any point (τ_0', x_0') with τ_0' close enough to b (resp. a) by means of a geodesic $(\tau(t), \gamma_F(t))$ with $\mathrm{d}\tau/\mathrm{d}t > 0$ (resp. < 0) near τ_0'. Condition (B) is weaker than Condition (A), and implies not only geodesic connectedness but also that if Condition (A) does not hold at b (resp. a) then any $(\tau_0, x_0) \in I \times F$ can be joined with a point (τ_0', x_0') with τ_0' close enough to b (resp. a) by means of a geodesic with $\epsilon = 1$ (resp. $\epsilon = -1$), and perhaps using Convention 1 once close to τ_0'. Condition (C) is the most general condition for geodesic connectedness, which just drops a residual case covered by Condition (R).

Definition 1. Let $f : (a, b) \to \mathbb{R}$ be a smooth function and let $m_b = \liminf_{\tau \to b} f(\tau)$ (resp. $m_a = \liminf_{\tau \to a} f(\tau)$). The extreme b (resp. a) is a (strict) relative minimum of f if

1. when $b < \infty$ (resp. $a > -\infty$), there exists $\epsilon > 0$ such that if $0 < \epsilon' < \epsilon$, then $f(b - \epsilon') > m_b$ (resp. $f(a + \epsilon') > m_a$);
2. when $b = \infty$ (resp. $a = -\infty$), there exist $M > 0$ such that if $M' > M$ then $f(M') > m_b$ (resp. $f(-M') > m_a$).

Condition (A) for f. Either $1/f^2$ does not reach at b (resp. a) a relative minimum in the sense of Definition 1 or, otherwise

$$\int_c^b f^{-2} \left(\frac{1}{f^2} - m_b \right)^{-1/2} = \infty \qquad \left(\text{resp. } \int_a^c f^{-2} \left(\frac{1}{f^2} - m_a \right)^{-1/2} = \infty \right)$$

for some $c \in (a, b)$ close to b (resp. a), i.e. $c \in (b - \epsilon, b)$ (resp. $c \in (a, a + \epsilon)$) if the extreme b (resp. a) is finite or $c > M$ (resp. $c < -M$) if this extreme is infinite, where ϵ and M are given in Definition 1.

The following definition is needed to state Condition (B). Recall that this definition is applicable just when Condition (A) does not hold.

Definition 2. Assume that the function $1/f^2$ reaches at b (resp. at a) a relative minimum such that $\int_c^b f^{-2}(1/f^2 - m_b)^{-1/2} < \infty$ (resp. $\int_a^c f^{-2}(1/f^2 - m_a)^{-1/2} < \infty$) for some $c \in (a, b)$. Then we define

$$d_b = \limsup_{D \to m_b} \left(\int_c^{b_*} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right) - \int_c^b f^{-2} \left(\frac{1}{f^2} - m_b \right)^{-1/2}$$

$$\left(\text{resp.} \quad d_a = \limsup_{D \to m_a} \left(\int_{a_*}^{c} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right) - \int_a^c f^{-2} \left(\frac{1}{f^2} - m_a \right)^{-1/2} \right),$$

where $b^* \equiv b^*[D]$ (resp. $a^* \equiv a^*[D]$) is given by (11).

Remark.

1. *Note that the uniform convergence of $f^{-2}(1/f^2 - D)^{-1/2}$ on compact subsets of (a, b) when D varies, ensures that d_b and d_a are independent of c.*
2. *It is easy to check that $d_b,\ d_a \geq 0$. As when $D \to m_b$ then $b^* \to b$, where b is a relative minimum, it is clear that if there were continuity of the integrals with D at m_b (resp. m_a) then $d_b = 0$ (resp. $d_a = 0$). But as we will see in the example below, there exist cases in which the inequalities are strict, and $d_b,\ d_a$ can reach even the value ∞.*

Condition (B) for f. Either $1/f^2$ does not reach at b (resp. a) a relative minimum or, otherwise, it verifies either $\int_c^b f^{-2}(1/f^2 - m_b)^{-1/2} = \infty$ for some $c \in (a, b)$ as in Condition (A), or $2d_b \geq \operatorname{diam}(F) \in (0, \infty]$ (resp. either $\int_a^c f^{-2}(1/f^2 - m_a)^{-1/2} = \infty$ or $2d_a \geq \operatorname{diam}(F) \in (0, \infty]$).

Obviously Condition (A) implies Condition (B), but the converse is not true as the following example shows.

Example. Consider the function $1/g^2(\tau) = 1 - \tau$ defined on $(0, 1)$. Modify this function smoothly on $\{I_n\}_{n \in \mathbb{N}}$, $I_n = (a_n, b_n)$, $a_n, b_n \to 1$, $a_n < b_n < a_{n+1}$ in such a way that the modified function $1/f^2$ satisfies $1/f^2 > 1/g^2$ on I_n $\forall n \in \mathbb{N}$ and

$$\int_0^{b_{n*}} f^{-2} \left(\frac{1}{f^2} - D_n \right)^{-1/2} \geq 2L, \tag{12}$$

where $\int_0^1 f^{-1} = L$ and D_n is chosen decreasing to 0 and such that $b_{n*} = \frac{1}{2}(a_n + b_n)$; this is possible by taking $1/f^2$ with derivative small enough in $(a_n, \frac{1}{2}(a_n + b_n))$ (e.g. if this derivative vanishes at $\frac{1}{2}(a_n + b_n)$ the integral (12) will be infinite). Then, as $d_b \geq L$, it is sufficient to take (F, g) such that $2d_b \geq \operatorname{diam}(F)$ (see Fig. 1).

Lemma 3. *If Condition (B) does not hold at b (resp. a) then there exist* $\lim_{\tau \to b} f \in (0, \infty]$ (*resp.* $\lim_{\tau \to a} f \in (0, \infty]$) *and* $f' > 0$ *on* $(b - \delta, b)$ *or* (M, ∞) (*resp.* $f' < 0$ *on* $(a, a + \delta)$ *or* $(-\infty, -M)$) *for some* $\delta > 0$ *small or* $M > 0$ *big.*

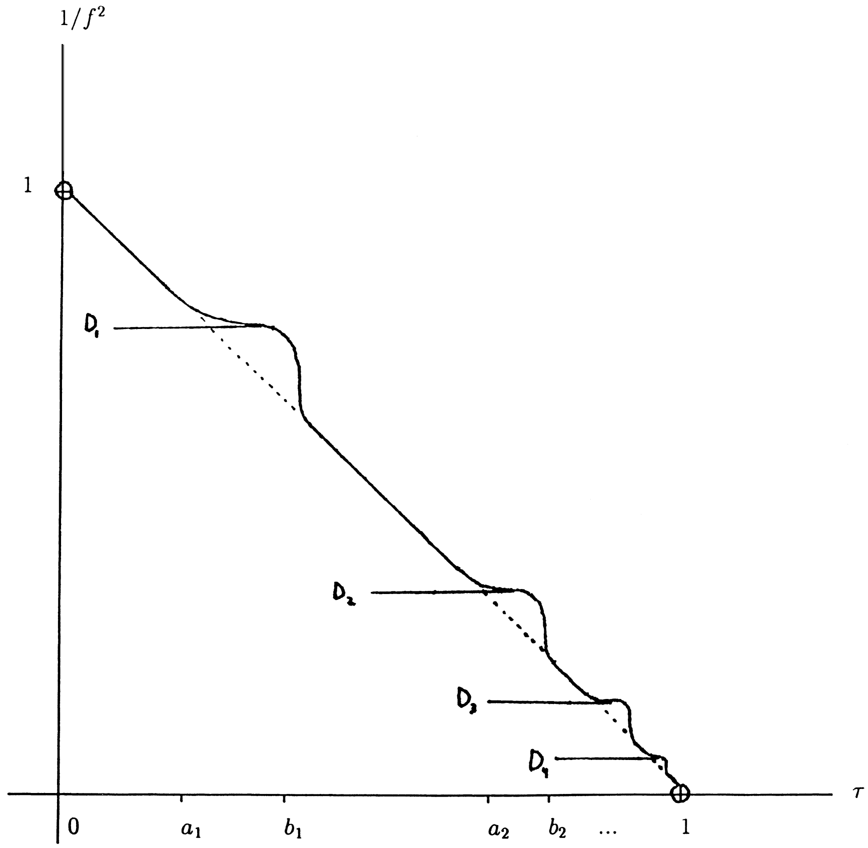

Fig. 1. Condition (B) is satisfied but not Condition (A).

Proof. Reasoning for $b < \infty$, assume that Condition (B) does not hold at b. Then $1/f^2$ reaches a relative minimum at b and $\int_{\tau_0}^{b} f^{-2}(1/f^2 - m_b)^{-1/2} < \infty$ for certain $\tau_0 \in I$ (see Definition 1). It is sufficient to prove that $f' > 0$ on $(b - \delta, b)$. Otherwise, there exist a sequence $\{\overline{\tau}_n\}_{n\in\mathbb{N}}$, $\tau_0 < \overline{\tau}_n \in I, \overline{\tau}_n \to b$ such that $f'(\overline{\tau}_n) \leq 0$. If we choose a maximum τ_n of f on $[\tau_0, \overline{\tau}_n]$, then $f'(\tau_n) = 0$ for n big enough. Thus, $\int_{\tau_0}^{b^*} f^{-2}(1/f^2 - D_n)^{-1/2} = \infty$ for $D_n = 1/f^2(\tau_n)$. The choice of τ_n implies that $D_n \to m_b$, which contradicts that $d_b < \infty$. □

Remark. *If Condition* (*B*) *does not hold at b* (*resp. a*) *then* $\lim_{\tau\to b} 1/f^2 = m_b$ (*resp.* $\lim_{\tau\to a} 1/f^2 = m_a$).

From Lemma 3, it is natural to construct Table 1, where it is assumed that f is continuously extendible to b (the table for a would be analogous, but reversing the sign of the corresponding β).

The following definition, necessary to state Condition (C), is applicable when Condition (B) does not hold.

Definition 3. Assume that the function $1/f^2$ reaches at b (resp. a) a relative minimum such that $2d_b < \operatorname{diam}(F)$ and $m < m_b$ (resp. $2d_a < \operatorname{diam}(F)$ and $m < m_a$) where m is the infimum value of $1/f^2$ in (a, b). Choose $\tau_0 \in (b-\epsilon, b)$ or $\tau_0 > M$ (resp. $\tau_0 \in (a, a+\epsilon)$ or $\tau_0 < -M$) where ϵ, M are given in Definition 1. Then we define

$$i_b = \inf_{D\in(m,m_b]} \left\{ \int_{a_*}^{b} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\}$$

$$\left(\text{resp.} \quad i_a = \inf_{D\in(m,m_a]} \left\{ \int_{a}^{b_*} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\} \right),$$

where $b^* \equiv b^*[D]$ (resp. $a^* \equiv a^*[D]$) is given by (11).

Note that this definition is independent of the choice of τ_0.

Condition (C). Either $1/f^2$ does not reach at b (resp. a) a relative minimum or, otherwise either $\int_c^b f^{-2}(1/f^2 - m_b)^{-1/2} = \infty$ for some $c \in (a, b)$ as in Condition (A), or $2d_b \geq \operatorname{diam}(F)$, or $d_b \geq i_b$ (resp. either $\int_a^c f^{-2}(1/f^2 - m_a)^{-1/2} = \infty$ or $2d_a \geq \operatorname{diam}(F)$ or $d_a \geq i_a$).

Again Condition (B) implies obviously Condition (C), and a counterexample to the converse is shown.

Example. Let $1/f^2$ be the function in the previous example. We have that the smooth function $\bar{f}$ defined on $(-1/N, 1)$ such that $\lim_{\tau\to -1/N} 1/\bar{f}^2 = 0$, $1/\bar{f}^2(0) = N+1$ and $1/\bar{f}^2(\tau) = N + 1/f^2(\tau)$ for $\tau \in (0, 1)$ satisfies that $i_b \leq d_b$ for N big enough. Then it is sufficient to take (F, g) such that $2d_b < \operatorname{diam}(F)$ (see Fig. 2).

For the remaining residual case, we need the following definition, where Convention 1 is explicitly used.

Definition 4. Assume $1/f^2 > m$ for $\tau \in (a, b)$ and $m_a = m_b = m$. Then we define

$$r_i^n(\tau_0) = \lim_{\epsilon\searrow 0} \liminf_{D\searrow m} \left\{ {}_{(-1)^n[n-1]} \int_{\tau_0}^{a_*} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right.$$

$$\left. + \int_{a_*}^{b-\epsilon} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\},$$

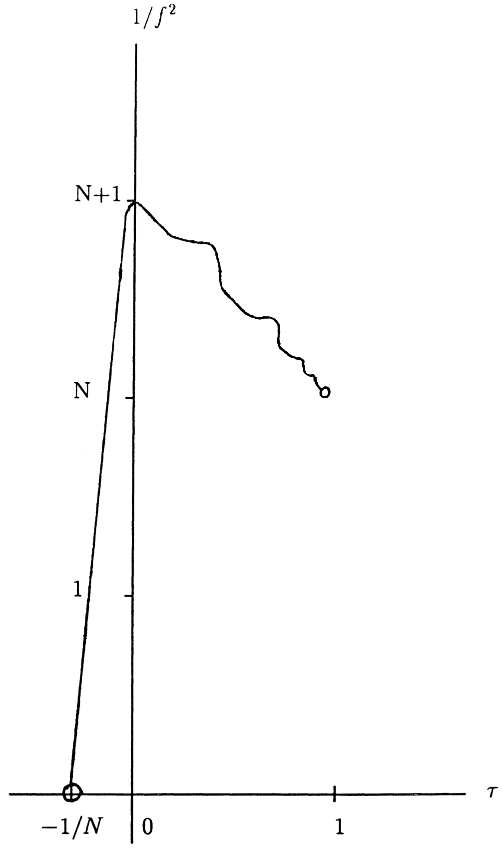

Fig. 2. Condition (C) is satisfied but not Condition (B).

$$r_s^n(\tau_0) = \lim_{\epsilon\searrow 0}\limsup_{D\searrow m}\left\{ {}_{(-1)^n[n]}\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2} + \int_{b-\epsilon}^{b_*} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2}\right\},$$

$$l_i^n(\tau_0) = \lim_{\epsilon\searrow 0}\liminf_{D\searrow m}\left\{ {}_{(-1)^{n-1}[n-1]}\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2} + \int_{a+\epsilon}^{b_*} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2}\right\},$$

$$l_s^n(\tau_0) = \lim_{\epsilon\searrow 0}\limsup_{D\searrow m}\left\{ {}_{(-1)^{n-1}[n]}\int_{\tau_0}^{a_*} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2} + \int_{a_*}^{a+\epsilon} f^{-2}\left(\frac{1}{f^2}-D\right)^{-1/2}\right\} \quad (13)$$

for $n \geq 1$, and

$$r_i^0(\tau_0) = \lim_{\epsilon \searrow 0} \liminf_{D \searrow m} \left\{ \int_{\tau_0}^{b-\epsilon} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\},$$
$$r_s^0(\tau_0) = \lim_{\epsilon \searrow 0} \limsup_{D \searrow m} \left\{ \int_{\tau_0}^{b_*} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} + \int_{b-\epsilon}^{b_*} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\},$$
$$l_i^0(\tau_0) = \lim_{\epsilon \searrow 0} \liminf_{D \searrow m} \left\{ \int_{a+\epsilon}^{\tau_0} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\},$$
$$l_s^0(\tau_0) = \lim_{\epsilon \searrow 0} \limsup_{D \searrow m} \left\{ \int_{a_*}^{\tau_0} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} + \int_{a_*}^{a+\epsilon} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} \right\}. \tag{14}$$

If some extreme of I is infinite, previous definition must be understood in the natural way (see comments above formula (27)).

Recall that $r_i^0(\tau_0) = \int_{\tau_0}^{b} f^{-2}(1/f^2 - m)^{-1/2}$ (resp. $l_i^0(\tau_0) = \int_a^{\tau_0} f^{-2}(1/f^2 - m)^{-1/2}$). It is also clear that $r_i^n(\tau_0) \leq r_s^n(\tau_0)$ (resp. $l_i^n(\tau_0) \leq l_s^n(\tau_0)$) and the sequence $\{r_i^n(\tau_0)\}_{n\in\mathbb{N}}$ is strictly increasing to ∞ (resp. replacing i or r by s or l).

Condition (R). Assume $1/f^2 > m$ for $\tau \in (a, b)$ and $m_a = m_b = m$, then $[r_i^0(\tau_0), \mathrm{diam}(F)] \subseteq \cup_{n\geq 0}[r_i^n(\tau_0), r_s^n(\tau_0)]$ and $[l_i^0(\tau_0), \mathrm{diam}(F)] \subseteq \cup_{n\geq 0}[l_i^n(\tau_0), l_s^n(\tau_0)]$ for every $\tau_0 \in I$.

Remark 2. *When $1/f^2 > m$ for $\tau \in (a, b)$ and $m_a = m_b = m$, it is clear that Condition (C) holds if and only if Condition (B) holds; moreover, Condition (R) is less restrictive than Condition (B). In fact, when Condition (A) holds then $r_i^0(\tau_0) = \infty = l_i^0(\tau_0)$ for all $\tau_0 \in I$, thus Condition (R) is automatically satisfied. When Condition (A) does not hold then if $2d_b \geq \mathrm{diam}(F)$ (i.e. Condition (B) holds at b) then $r_s^0(\tau_0) \geq \mathrm{diam}(F)$ for all $\tau_0 \in I$ (and, thus Condition (R) holds).*

Condition (C) and Condition (R) provide us accurate sufficient hypotheses for geodesic connectedness, as the following two theorems show. (For the sake of completeness, we also state the result on connection by causal geodesics, already contained in [17, Theorems 3.3 and 3.7].

Theorem 1. *Let $(I \times F,\ g^f = -\mathrm{d}\tau^2 + f^2 g)$ be a GRW space–time with weakly convex fiber (F, g). Then*

1. *Two points $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0')$, $\tau_0 < \tau_0'$ are chronologically (resp. causally) related if and only if $\int_{\tau_0}^{\tau_0'} f^{-1} > d_F(x_0, x_0')$ (resp. $\geq d_F(x_0, x_0')$) and, in this case, they can be joined with at least one timelike (resp. non-spacelike) geodesic.*
2. *If Condition (C) or Condition (R) holds then the GRW space–time is geodesically connected.*

When the fiber is strongly convex, Condition (C) or Condition (R) becomes also necessary.

Theorem 2. *Let* $(I \times F,\ g^f = -\mathrm{d}\tau^2 + f^2 g)$ *be a GRW space–time with strongly convex fiber* (F, g). *Then*

1. *Each two causally related points can be joined with exactly one* (*necessarily non-spacelike*) *geodesic*.
2. *The GRW space–time is geodesically connected if and only if either Condition* (*C*) *or Condition* (*R*) *holds*.

From its proof, the naturality of the strong convexity assumption is clear. However, we discuss below the proof of Theorem 2, what happens if just weak convexity is assumed.

As a consequence of our technique, we also obtain the following result on multiplicity.

Theorem 3. *Let* $(I \times F,\ g^f = -\mathrm{d}\tau^2 + f^2 g)$ *be a GRW space–time with weakly convex fiber* (F, g) *and assume that either Condition* (*A*) *or Condition* (*B*) *with* d_a, d_b (*if defined*) *equal to infinity, holds*.

Then there exist a natural surjective map between geodesics connecting $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0') \in I \times F$ *and F-geodesics connecting* x_0 *and* x_0'.

Moreover, if (F, g) *is complete and F is not contractible in itself then any* $z_0, z_0' \in I \times F$ *can be joined by means of infinitely many spacelike geodesics. If the corresponding* x_0, x_0' *are not conjugate in* (F, g), *then there are at most finitely many causal geodesics connecting* z_0, z_0' *in* $I \times F$.

Remark. *From results in Section* 5, *it will be clear that to impose the non-conjugacy of* x_0, x_0' *as above is less restrictive than to impose the non-conjugacy of* z_0, z_0'. *On the other hand, the completeness of the fiber in Theorem* 3 *can be replaced for a convexity assumption of the Cauchy boundary* [2].

4. Proof of theorems

Consider a GRW space–time $(I \times F,\ -\mathrm{d}\tau^2 + f^2 g)$ with weakly convex fiber (F, g). For fixed $\tau_0 \in I$ put

$$m_r = \mathrm{Inf}\left\{ \frac{1}{f^2(\tau)} \middle| \tau \in [\tau_0, b) \right\}, \qquad m_l = \mathrm{Inf}\left\{ \frac{1}{f^2(\tau)} \middle| \tau \in (a, \tau_0] \right\}. \tag{15}$$

Lemma 4. *Using the notation* (11), *the function in D*

$$\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2}, \quad b_* \equiv b_*(D)$$

$$\left(\text{resp.} \quad \int_{a_*}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2}, \quad a_* \equiv a_*(D)\right)$$

with values in $(0, \infty]$ *is continuous when D varies in* $(m_r, 1/f^2(\tau_0))$ (*resp.* $(m_l, 1/f^2(\tau_0))$).

Proof. We will check that every convergent sequence $\{D^k\}_{k\in\mathbb{N}}$, $D^k \to D^\infty$, $D^\infty \in (m_r, 1/f^2(\tau_0))$ satisfies $\int_{\tau_0}^{b_*^k} f^{-2}(1/f^2 - D^k)^{-1/2} \to \int_{\tau_0}^{b_*^\infty} f^{-2}(1/f^2 - D^\infty)^{-1/2}$ (the case with a_* is analogous). We can consider the following possibilities:

1. If $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{b_*^\infty} \neq 0$ then the sequence of intervals $[\tau_0, b_*^k)$ converges to $[\tau_0, b_*^\infty)$ and the integrands converge uniformly on $[\tau_0, b_*^\infty - \delta]$ for $\delta > 0$ small, which implies the convergence of the integrals in $[\tau_0, b_*^\infty - \delta]$. Thus, the result follows because the integrals on $[b_*^\infty - \delta, b_*^\infty]$ goes to zero when $\delta \to 0$.
2. If $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{b_*^\infty} = 0$ then the uniform convergence of $f^{-2}(1/f^2 - D^k)^{-1/2}$ to $f^{-2}(1/f^2 - D^\infty)^{-1/2}$ on compact subsets of $[\tau_0, b_*^\infty)$ implies that $\int_{\tau_0}^{b_*^k} f^{-2}(1/f^2 - D^k)^{-1/2} \to \infty = \int_{\tau_0}^{b_*^\infty} f^{-2}(1/f^2 - D^\infty)^{-1/2}$. □

Recall that the integrals not necessarily vary continuously when $D = m_r, m_l$.

In what follows we will use the function $\tau(K)$ defined in Remark 1, and follow the notation: $\tau^- = \tau(K^-)$, $\tau^+ = \tau(K^+)$.

Lemma 5. *Consider* $(\tau_0, x_0) \in I \times F$ *and* $x_0' \in F$ *such that* $d(x_0, x_0') = L > 0$. *The function* $\tau(K)$ *is continuous on its domain* $\mathcal{D}$. *Moreover, if* $\mathrm{d}/\mathrm{d}\tau|_{\tau=\tau_0} 1/f^2(\tau) = 0$ *then* $\tau(K)$ *can be continuously extended to* $K = 0$ *by* $\tau(0) = \tau_0$.

As a consequence, if $[K^-, K^+] \subset \mathcal{D}$ *then we can connect* (τ_0, x_0) *with* $[\tau^-, \tau^+] \times \{x_0'\}$ (*or* $[\tau^+, \tau^-] \times \{x_0'\}$).

Proof. Firstly, we will check that every convergent sequence $\{K^n\}_{n\in\mathbb{N}}$, $K^n \to K^\infty > 0$ (< 0 analogous), $K^n, K^\infty \in \mathcal{D}$ for all n, satisfies that $\tau^n \to \tau^\infty$, where $\tau^n = \tau(K^n)$, $\tau^\infty = \tau(K^\infty)$. Assume first $K^\infty \neq 0$, then

1. If $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{b_*^\infty, a_*^\infty} \neq 0$, then easily $a_*^n \to a_*^\infty$, $b_*^n \to b_*^\infty$, so the proof follows from Lemma 4.
2. If $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{b_*^\infty} = 0$ then, as $\int_{\tau_0}^{b_*^\infty} f^{-2}(1/f^2 - D^\infty)^{-1/2} = \infty$, we have $\tau^\infty < b_*^\infty$ and the uniform convergence of the integrand on a compact set $[\tau_0, \tau^\infty + \delta]$ ($\delta > 0$ small) proves the result.
3. If $b_*^\infty = b$ then again $\tau^\infty < b$ and the result follows from the convergence on $[\tau_0, \tau^\infty + \delta]$.

The remaining cases follow from combinations of the previous ones.

Now, consider the case that $K^\infty = 0 \in \mathcal{D}$ and (necessarily) $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{\tau_0} \neq 0$. Then it is easy to check that Lemma 4 can be extended to $D = 1/f^2(\tau_0)$, which implies the continuity of τ at 0.

So, we have just to prove that if $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{\tau_0} = 0$, then $\tau(K)$ can be continuously extended as $\tau(0) = \tau_0$. For fixed $\epsilon > 0$, the limit of $\int_{\tau_0}^{\tau_0+\epsilon} f^{-2}(1/f^2 - D)^{-1/2}$ and $\int_{\tau_0-\epsilon}^{\tau_0} f^{-2}(1/f^2 - D)^{-1/2}$ (for the values of D where they are well defined) are ∞ when $D \nearrow 1/f^2(\tau_0)$ (and, thus, $K \to 0$), from which the result follows. □

Lemma 6. *If* $K^+ > 0$ (*resp.* $K^- < 0$) *belongs to the domain* $\mathcal{D}$ *of* $\tau(K)$ *but* $K^+ - \epsilon \geq 0$ (*resp.* $K^- + \epsilon \leq 0$) *for some* $\epsilon > 0$, *does not belong, then we can connect* (τ_0, x_0) *with* $(a, \tau^+] \times \{x_0\}$ (*resp.* $[\tau^-, b) \times \{x_0'\}$) *by means of geodesics with* $K \in (K^+ - \epsilon, K^+]$ (*resp.* $K \in [K^-, K^- + \epsilon)$).

Proof. Reasoning for K^+, define $K_0 = \inf\{K \leq K^+ : [K, K^+] \subseteq \mathcal{D}\}$. As $0 \leq K_0 < K^+$, the fact that K_0 is the infimum implies that $b_*(D_0) \neq b$ where $D_0 = 1/f^2(\tau_0) - K_0$. Therefore, $\lim_{K \searrow K_0} \tau(K) = a$ (otherwise, it would contradict that K_0 is the infimum again) and the result follows from the first assertion in Lemma 5. □

Lemma 7. *If the domain $\mathcal{D}$ contains $K^+ > 0$ and $K^- < 0$, and the inequality $\tau^- < \tau^+$ holds, then we can connect (τ_0, x_0) with, at least $[\tau^-, \tau^+] \times \{x'_0\}$ by choosing $K \in [K^-, K^+]$.*

Proof. If τ is defined in $[K^-, K^+]$ then Lemma 5 can be applied. Otherwise, let $K_0 \in (K^-, K^+)$ be such that $K_0 \notin \mathcal{D}$. If, say $K_0 \geq 0$ Lemma 6 can be applied to K^+. □

Now, a first result on geodesic connectedness can be stated.

Lemma 8. *A GRW space–time $(I \times F, -\mathrm{d}\tau^2 + f^2 g)$ with weakly convex fiber (F, g) and satisfying Condition (A) is geodesically connected.*

Proof. Let $(\tau_0, x_0), (\tau'_0, x'_0) \in I \times F$, $L = d(x_0, x'_0)$, $L > 0$. We consider the following cases according to the values of m_l, m_r in (15):

1. Case $m_l, m_r < 1/f^2(\tau_0)$. Then $\int_{\tau_0}^{b_*} f^{-2}(1/f^2 - m_r)^{-1/2} = \infty$, $\int_{a_*}^{\tau_0} f^{-2}(1/f^2 - m_l)^{-1/2} = \infty$ and, thus, there exist $a_* < \tau^- < \tau_0 < \tau^+ < b_*$ such that $\int_{\tau_0}^{\tau^+} f^{-2}(1/f^2 - m_r)^{-1/2} = L$, $\int_{\tau^-}^{\tau_0} f^{-2}(1/f^2 - m_l)^{-1/2} = L$; so (τ_0, x_0) can be joined with $(\tau_\pm, x'_0)$. By using Lemma 7 we can connect (τ_0, x_0) with $[\tau^-, \tau^+] \times \{x'_0\}$ taking $K \in [K^-, K^+]$. Moreover, fixed $\epsilon > 0$ such that $\tau^+ + \epsilon < b$ (resp. $\tau^- - \epsilon > a$) the limit of $\int_{\tau_0}^{\tau^+ + \epsilon} f^{-2}(1/f^2 - D)^{-1/2}$ (resp. $\int_{\tau^- - \epsilon}^{\tau_0} f^{-2}(1/f^2 - D)^{-1/2}$) is greater than L when $D \to m_r$ (resp. $D \to m_l$) and the limit is 0 when $D \to -\infty$; so (τ_0, x_0) can be connected with $(\tau^+ + \epsilon, x'_0)$ and $(\tau^- - \epsilon, x'_0)$. Therefore, we can also connect (τ_0, x_0) with $[\tau^+, b) \times \{x'_0\}$ and $(a, \tau^-] \times \{x'_0\}$ taking $K \in [K^+, \infty)$ and $K \in (-\infty, K^-]$, respectively. In particular $(\tau_0, x_0), (\tau'_0, x'_0)$ can be joined.
2. Case $m_l = m_r = 1/f^2(\tau_0)$. Assume, say $\tau_0 < \tau'_0$, then $\int_{\tau_0}^{\tau'_0} f^{-2}(1/f^2 - D)^{-1/2}$ goes to 0 if $D \to -\infty$ and to ∞ if $D \nearrow 1/f^2(\tau_0)$. Therefore, there exist $D^* < 1/f^2(\tau_0)$ such that $\int_{\tau_0}^{\tau'_0} f^{-2}(1/f^2 - D^*)^{-1/2} = L$ and the proof is over.
3. Case $m_l = 1/f^2(\tau_0)$ and $m_r < 1/f^2(\tau_0)$ (the remaining case is analogous). If, for certain $\delta > 0$, $\int_a^{\tau_0} f^{-2}(1/f^2 - m_l - \delta)^{-1/2} > L$ then we can follow an argument as in (1). Otherwise, let τ^+ be such that $\int_{\tau_0}^{\tau^+} f^{-2}(1/f^2 - m_r)^{-1/2} = L$. Fixed $\epsilon > 0$, the limit of $\int_{\tau_0}^{\tau^+ + \epsilon} f^{-2}(1/f^2 - D)^{-1/2}$ is 0 when $D \to -\infty$ and it is greater than L when $D \to m_r$; thus, we can connect (τ_0, x_0) with $(\tau^+ + \epsilon, x'_0)$ and, therefore, with $[\tau^+, b) \times \{x'_0\}$ by means of geodesics with $K \in [K^+, \infty)$. Finally, from Lemma 6, we can also connect (τ_0, x_0) with $(a, \tau^+] \times \{x'_0\}$ taking $K \in (0, K^+]$. □

Lemma 9. *A GRW space–time $(I \times F, -\mathrm{d}\tau^2 + f^2 g)$ with weakly convex fiber (F, g) and satisfying Condition (B) is geodesically connected.*

Proof. Let $(\tau_0, x_0), (\tau_0', x_0') \in I \times F$, $L = d(x_0, x_0')$, $L > 0$ be. Firstly, suppose the case

$$\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2} - m_r\right)^{-1/2} \leq L, \qquad \int_{a_*}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - m_l\right)^{-1/2} \leq L, \tag{16}$$

$m_r = m_b$, $m_l = m_a$ and $2d_b \geq L$, $2d_a \geq L$.

Note that

$$\frac{1}{f^2(\tau_0)} \geq \mathrm{Max}\{m_a, m_b\}, \tag{17}$$

and we consider first that this inequality is strict. Then, for fixed $\delta > 0$ such that $a+\delta < \tau_0, \tau_0'$ and $\tau_0, \tau_0' < b - \delta$, there exist $0 < K_\delta^r < 1/f^2(\tau_0) - m_b$ and $m_a - 1/f^2(\tau_0) < K_\delta^l < 0$ such that $\tau(K_\delta^r) > b - \delta$ and $\tau(K_\delta^l) < a + \delta$; recall that, otherwise, say

$$2\int_{b-\delta}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} < L \leq 2d_b$$
$$= 2\limsup_{\hat{D}\to m_b}\left(\int_{b-\delta}^{b_*} f^{-2}\left(\frac{1}{f^2} - \hat{D}\right)^{-1/2}\right) - 2\int_{b-\delta}^{b} f^{-2}\left(\frac{1}{f^2} - m_b\right)^{-1/2}$$

for all $D > m_b$ (with $b_*(D) > b - \delta$), which is a contradiction because $\int_{b-\delta}^{b} f^{-2}(1/f^2 - m_b)^{-1/2} > 0$.

So, the geodesics corresponding to K_δ^r and K_δ^l join (τ_0, x_0) with (τ^r, x_0') and (τ^l, x_0'), where $\tau^r = \tau(K_\delta^r)$, $\tau^l = \tau(K_\delta^l)$. From Lemma 7, we can connect (τ_0, x_0) with $[\tau^l, \tau^r] \times \{x_0'\}$ taking $K \in [K_\delta^l, K_\delta^r]$ and, thus, the connectedness of (τ_0, x_0) with (τ_0', x_0') is obtained.

If (17) holds with equality, then because of (16) we have $m_a \neq m_b$ (say $m_a > m_b$), and $K = 0$ does not belong to the domain $\mathcal{D}$ of $\tau(K)$. Reasoning as above $K^+ \in \mathcal{D}$, $K^+ > 0$ is found, and the result follows from Lemma 6.

Finally, the remaining cases (where not necessarily both inequalities (16) hold) are combinations of this one and the cases in Lemma 8. □

Now, we are ready to prove our main result on connectedness. The proof of (2) in Theorem 1 is the consequence of Propositions 1 and 2.

Proposition 1. *Let $(I \times F, -\mathrm{d}\tau^2 + f^2 g)$ be a GRW space–time with weakly convex fiber (F, g) and satisfying Condition (C). Then it is geodesically connected.*

Proof. Let $(\tau_0, x_0), (\tau_0', x_0') \in I \times F$, $L = d(x_0, x_0')$, $L > 0$ be. Suppose

$$\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2} - m_r\right)^{-1/2} \leq L, \qquad \int_{a_*}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - m_l\right)^{-1/2} \leq L,$$

$m_a = m_l < m_r = m_b$, $2d_b < L \leq \mathrm{diam}(F)$, $2d_a \geq L$ and $d_b \geq i_b$ (from Lemma 9 this is the only relevant case to study). As $d_b \geq i_b$ there exist $D_1^r \leq m_b$ such that

$$\int_{a_*}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - D_1^r\right)^{-1/2} + \int_{a_*}^{b} f^{-2}\left(\frac{1}{f^2} - D_1^r\right)^{-1/2} < 2d_b < L.$$

On the other hand, as $2d_a \geq L$ for $D_2^r < D_1^r$ near enough to m_l we have

$$\int_{a_*}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - D_2^r\right)^{-1/2} + \int_{a_*}^{b} f^{-2}\left(\frac{1}{f^2} - D_2^r\right)^{-1/2} > L.$$

Therefore, the domain $\mathcal{D}$ of $\tau(K)$ contains $K_2^r = D_2^r - 1/f^2(\tau_0)$ but not $K_1^r = D_1^r - 1/f^2(\tau_0)$. From Lemma 6, (τ_0, x_0) can be connected with $[\tau(K_2^r), b) \times \{x_0'\}$. Choose $K^r = D^r - 1/f^2(\tau_0) \in [K_2^r, K_1^r)$ such that $\tau(K^r) > b - \delta$ for δ small. As $2d_a \geq L$, there exist D^l, $m_a < D^l < D^r$ such that $\tau(K^l) < a + \delta$ $(K^l = D^l - 1/f^2(\tau_0))$. Thus, the result follows from Lemma 7. □

Proposition 2. *Let $(I \times F, -\mathrm{d}\tau^2 + f^2 g)$ be a GRW space–time with weakly convex fiber (F, g) and satisfying Condition (R). Then it is geodesically connected.*

Proof. We will use systematically that if D is close enough to m and $D > m$ then $K^+ = 1/f^2(\tau_0) - D(> 0)$ and $K^- = D - 1/f^2(\tau_0)(< 0)$ satisfy $[K^-, K^+] \subset \mathcal{D}$; thus, Lemma 5 can be claimed. Let $(\tau_0, x_0), (\tau_0', x_0') \in I \times F$, $L = d(x_0, x_0')$, $L > 0$, and consider the following two cases:

1. Suppose $r_i^n(\tau_0) \leq L \leq r_s^n(\tau_0)$, $l_i^{n'}(\tau_0) \leq L \leq l_s^{n'}(\tau_0)$ for certain $n, n' \geq 0$. Fix $\epsilon > 0$ such that $a + \epsilon < \tau_0' < b - \epsilon$. Then for some D_i^r, D_s^r close to m, chosen such that $m < D_i^r < D_s^r$, we have

$$\begin{aligned} &{}_{(-1)^n[n-1]}\int_{\tau_0}^{a_*(D_i^r)} f^{-2}\left(\frac{1}{f^2} - D_i^r\right)^{-1/2} + \int_{a_*(D_i^r)}^{b-\epsilon} f^{-2}\left(\frac{1}{f^2} - D_i^r\right)^{-1/2} < L, \\ &{}_{(-1)^n[n]}\int_{\tau_0}^{b_*(D_s^r)} f^{-2}\left(\frac{1}{f^2} - D_s^r\right)^{-1/2} + \int_{b-\epsilon}^{b_*(D_s^r)} f^{-2}\left(\frac{1}{f^2} - D_s^r\right)^{-1/2} > L, \end{aligned} \tag{18}$$

   if $n \geq 1$, or

$$\begin{aligned} &\int_{\tau_0}^{b-\epsilon} f^{-2}\left(\frac{1}{f^2} - D_i^r\right)^{-1/2} < L, \\ &\int_{\tau_0}^{b_*(D_s^r)} f^{-2}\left(\frac{1}{f^2} - D_s^r\right)^{-1/2} + \int_{b-\epsilon}^{b_*(D_s^r)} f^{-2}\left(\frac{1}{f^2} - D_s^r\right)^{-1/2} > L, \end{aligned} \tag{19}$$

   if $n = 0$. Reasoning similarly to the left, we obtain analogous D_i^l, D_s^l, with $m < D_i^l < D_s^l$. From Lemma 4, there exist D^r, D^l, with $D_i^r < D^r < D_s^r$, $D_i^l < D^l < D_s^l$ such that $\tau(K^r) > b - \epsilon$, $\tau(K^l) < a + \epsilon$, where $K^r = (-1)^n(1/f^2(\tau_0) - D^r)$, $K^l = (-1)^{n'-1}(1/f^2(\tau_0) - D^l)$. Therefore, as $a + \epsilon < \tau_0' < b - \epsilon$, the connectedness of (τ_0, x_0) with (τ_0', x_0') is a consequence of Lemma 5.
2. Suppose now $L < r_i^0(\tau_0)(< r_i^n(\tau_0))$ and $L < l_i^0(\tau_0)(< l_i^n(\tau_0))$. As we saw in Definition 4 and the comments below, $r_i^0(\tau_0) = \int_{\tau_0}^{b} f^{-2}(1/f^2 - m)^{-1/2}$ (analogously

for l_i^0), thus, there exist $\epsilon > 0$, $a + \epsilon < \tau_0' < b - \epsilon$ such that

$$\int_{\tau_0}^{b-\epsilon} f^{-2} \left(\frac{1}{f^2} - m \right)^{-1/2} > L, \qquad \int_{a+\epsilon}^{\tau_0} f^{-2} \left(\frac{1}{f^2} - m \right)^{-1/2} > L.$$

But the limit of $\int_{\tau_0}^{b-\epsilon} f^{-2}(1/f^2 - D)^{-1/2}$ when $D \to -\infty$ is 0, thus we obtain $D^r < m$ such that $\int_{\tau_0}^{b-\epsilon} f^{-2}(1/f^2 - D^r)^{-1/2} = L$. So, taking $K^r = 1/f^2(\tau_0) - D^r (> 0)$, we obtain $\tau(K^r) = b - \epsilon$. Analogously, there exist $K^l < 0$ such that $\tau(K^l) = a + \epsilon$. Therefore, we obtain the connectedness of (τ_0, x_0) with (τ_0', x_0') from Lemma 5 again.

The remaining cases are combinations of the previous ones. □

Proof of Theorem 2. For (1) assume that $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0')$ are causally related and $\tau_0 < \tau_0'$. From Theorem 1 there exist a non-spacelike geodesic $\gamma : \mathcal{J} \to I \times F$, $\gamma(t) = (\tau(t), \gamma_F(t))$ joining them. As (F, g) is strongly convex, necessarily

$$\int_{\tau_0}^{\tau_0'} f^{-2} \left(\frac{1}{f^2} - D_0 \right)^{-1/2} = d(x_0, x_0') \tag{20}$$

being $D_0 = g(\mathrm{d}\gamma/\mathrm{d}t, \mathrm{d}\gamma/\mathrm{d}t) \le 0$. But the integral $\int_{\tau_0}^{\tau_0'} f^{-2}(1/f^2 - D)^{-1/2}$ is strictly increasing with D for $D \le 0$; thus, γ is the only causal geodesic joining z_0 and z_0'. Moreover, when $D > 0$ the integral (possibly under Convention 1) is bigger than when $D = 0$; so, no spacelike geodesic joins z_0 and z_0'.

In order to prove (2) assume that neither Condition (C) nor Condition (R) hold and consider the following cases. In the first three cases we will assume that Condition (R) is not applicable, and Condition (C) does not hold at b (at a would be analogous). Recall that, from Lemma 3, $1/f^2$ is decreasing at b; in the first case b is a non-unique absolute minimum; in the second, b is the unique absolute minimum, which is simpler; in the third, b is not an absolute minimum, which compels to use properly the definition of i_b. In the fourth case, Condition (R) is applicable, but it does not hold (neither does Condition (C), see Remark 2).

1. Assume that b is a relative minimum of $1/f^2$ and $m = m_b$ is reached at a point $\tau_m \in (a, b)$. As $2d_b < \mathrm{diam}(F)$, choose $L > 0$ such that $2d_b < L < \mathrm{diam}(F)$. From this choice, there exist $\tau_0^r > \tau_m$, close to b such that

$$2\int_{\tau_0^r}^{b_*(D)} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} < L \quad \forall D \in \left(m, \frac{1}{f^2(\tau_0^r)} \right). \tag{21}$$

As τ_m is a minimum, $(\mathrm{d}/\mathrm{d}\tau)1/f^2|_{\tau_m} = 0$. Thus, there exist τ_0^l near enough to b such that

$$2\int_{a_*(D)}^{\tau_0^l} f^{-2} \left(\frac{1}{f^2} - D \right)^{-1/2} > L \quad \forall D \in \left(m, \frac{1}{f^2(\tau_0^l)} \right). \tag{22}$$

Now, taking any $\tau_0 > \mathrm{Max}\{\tau_0^r, \tau_0^l\}$, $\tau_0' > \tau_0$ and x_0, x_0' with $d(x_0, x_0') = L$, it is clear that (21) and (22) forbid to connect (τ_0, x_0), (τ_0', x_0') by means of a geodesic.

2. Assume that b is a relative minimum, $m = m_b < m_a$ and $1/f^2(\tau) > m$ for all $\tau \in (a, b)$. Then, necessarily $2d_b < \mathrm{diam}(F)$. Choose again τ_0^r such that (21) holds. Recall that we can impose now, additionally, that τ_0^r is the strict minimum of $1/f^2$ on $(a, \tau_0^r]$. So, clearly (τ_0^r, x_0) and (τ_0', x_0') cannot be joined by a geodesic, if $\tau_0^r < \tau_0'$ and $d(x_0, x_0') = L$.
3. Assume that b is a relative minimum and $m < m_b$. As $d_b < i_b$ there exist τ_0^l such that

$$2\int_{a_*(D)}^{\tau_0^l} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} \geq 2d_b + 2\epsilon \quad \forall D \in (m, m_b] \tag{23}$$

for some $\epsilon > 0$ such that $2d_b + 2\epsilon < \mathrm{diam}(F)$. From the continuity stated in Lemma 4, there exist $\delta > 0$ such that inequality (23) holds if the right member is replaced by $L = 2d_b + \epsilon$ for all $D \in (m, m_b + \delta]$.

Now, as in case (1) we can take $\tau_0 (= \tau_0^r) > \tau_0^l$ with $1/f^2(\tau_0) < m_b + \delta$ and such that (21) holds for all D. Thus, for any $\tau_0' > \tau_0$, we cannot connect (τ_0, x_0), (τ_0', x_0') by means of a geodesic, if $d(x_0, x_0') = L$.
4. Assume that $1/f^2(\tau) > m$ for $\tau \in (a, b)$ and $m_a = m_b = m$. Suppose that Condition (R) is not fulfilled by, say the r's, i.e. $r_s^n(\tau_0) < r_i^{n+1}(\tau_0)$ with $r_s^n(\tau_0) < \mathrm{diam}(F)$ for certain $n \geq 0$ and $\tau_0 \in I$ (see the comments below Definition 4). Fix $L = d(x_0, x_0')(\leq \mathrm{diam}(F))$ with $r_s^n(\tau_0) < L < r_i^{n+1}(\tau_0)$. These inequalities imply for $n \geq 1$ that there exist an $\epsilon > 0$ such that

$$\begin{aligned} &\liminf_{D\searrow m} \left\{ {}_{(-1)^k[k-1]}\int_{\tau_0}^{a_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} + \int_{a_*}^{b-\epsilon} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} \right\} > L, \\ &\limsup_{D\searrow m} \left\{ {}_{(-1)^{k'}[k']}\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} + \int_{b-\epsilon}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} \right\} < L \end{aligned} \tag{24}$$

for $k = n + 1$, $k' = n$ and, thus for all $k \geq n + 1$ and $k' \leq n$. But this implies that for some $\delta > 0$ with $b^*(D = m + \delta) > b - \epsilon$ if $m < D < m + \delta$ then

$$\begin{aligned} &{}_{(-1)^k[k-1]}\int_{\tau_0}^{a_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} + \int_{a_*}^{b-\epsilon} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} > L, \\ &{}_{(-1)^{k'}[k']}\int_{\tau_0}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} + \int_{b-\epsilon}^{b_*} f^{-2}\left(\frac{1}{f^2} - D\right)^{-1/2} < L \end{aligned} \tag{25}$$

for $k \geq n + 1$, $k' \leq n$ (there are analogous inequalities when $n = 0$). Therefore (τ_0, x_0) cannot be geodesically connected with (τ_0', x_0') if $\tau_0' > b - \epsilon$. □

Discussion. Next, we will see what happens if we assume just weak convexity in Theorem 2 and Condition (R) is applicable (a similar study could be done if Condition (C) is applicable instead). As a consequence, we will give a proof of the (well known) non-geodesic connectedness of de Sitter space–time. It should be noticed that previous proofs use the high degree of symmetry of this space–time [7,20]. In our proof we will see what is the exact role of this symmetry.

Fix $z_0 = (\tau_0, x_0) \in I \times F$, $x_0' \in F$, $x_0 \neq x_0'$ and $\epsilon > 0$. Put $r^0_{i,\epsilon}(\tau_0)$, $r^n_{s,\epsilon}(\tau_0)$, etc. equal to the quantities in Definition 4 but without taking the limit $\epsilon \to 0$ (the extension of this new definition when $a = -\infty$ or $b = \infty$ is obvious, see de Sitter space–time below). Now, consider

$$A_\epsilon = \cup_{n\geq 0}[r^n_{i,\epsilon}(\tau_0), r^n_{s,\epsilon}(\tau_0)] \cup [0, r^0_{i,\epsilon}(\tau_0)],$$
$$B_\epsilon = \cup_{n\geq 0}[l^n_{i,\epsilon}(\tau_0), l^n_{s,\epsilon}(\tau_0)] \cup [0, l^0_{i,\epsilon}(\tau_0)],$$

and also

$$\mathcal{L} = \{\text{length } (\hat{\gamma}_F) | \hat{\gamma}_F \text{ is an } F\text{-geodesic which joins } x_0 \text{ and } x_0'\} \subset (0, \infty).$$

From the proof of Theorem 2, z_0 can be joined with $[a + \epsilon_0, b - \epsilon_0] \times \{x_0'\}$ if

$$\mathcal{L} \cap A_\epsilon \cap B_\epsilon \neq \emptyset$$

for some $\epsilon < \epsilon_0$. Moreover, it is also clear that z_0 cannot be joined with the points in $(a, a + \epsilon_0) \times \{x_0'\} \cup (b - \epsilon_0, b) \times \{x_0'\}$ if

$$\mathcal{L} \cap A_{\epsilon_0} = \emptyset, \qquad \mathcal{L} \cap B_{\epsilon_0} = \emptyset. \tag{26}$$

For de Sitter space–time, $I = \mathbb{R}$, $f = \cosh$ and the fiber is the usual sphere of radius 1. Recall that when the interval I is not bounded, we must replace $b - \epsilon$ (if $b = \infty$) and $-(a + \epsilon)$ (if $a = -\infty$) by $M > 0$, and the limit $\epsilon \to 0$ must be replaced by $M \to \infty$. Take $z_0 = (0, x_0)$; by Definition 4 ($M \to \infty$) we have

$$r^n_i(0) = r^n_s(0) = \tfrac{1}{2}\pi + n\pi = l^n_i(0) = l^n_s(0). \tag{27}$$

For $M = 0$, the new definitions $r^0_{i,\epsilon}(0)$, $r^n_{s,\epsilon}(0)$ ($\epsilon \equiv \infty$) read

$$r^n_{i,\epsilon}(0) = n\pi = l^n_{i,\epsilon}(0), \qquad r^n_{s,\epsilon}(0) = (n+1)\pi = l^n_{s,\epsilon}(0). \tag{28}$$

Now, choose x_0' as the antipodal point of x_0, i.e.,

$$\mathcal{L} = \{(2n+1)\pi | n = 0, 1, 2, \dots\}.$$

From the two limit cases (27) and (28), it is clear that condition (26) is fulfilled for any $M > 0$. So z_0 cannot be joined by means of a geodesic with $(-\infty, 0) \times \{x_0'\} \cup (0, \infty) \times \{x_0'\}$.

Summing up, for de Sitter space-time, the "symmetries" of its warping function are essential in order to have enough "holes" in A_{ϵ_0} and B_{ϵ_0}, where all the elements of $\mathcal{L}$ lie. But the only relevant symmetry of the fiber is that there are two points x_0, x_0' such that the lengths of the geodesics which join them has a constant gap. In our case, this gap (2π) and the symmetries of f fit well when $d(x_0, x_0') = \pi$.

Proof of Theorem 3. For the first assertion consider an F-geodesic $\hat{\gamma}_F(r)$, $r \in [0, L]$ with $L = \text{long}\, \hat{\gamma}_F$, $\hat{\gamma}_F(0) = x_0$ and $\hat{\gamma}_F(L) = x_0'$. From our hypotheses, if $1/f^2$ reaches a relative minimum at b (resp. a) and $b^*(m_r) = b$ (resp. $a^*(m_l) = a$) then

$$\text{either}\quad \int_{\tau_0}^{b^*(m_r)} f^{-2}\left(\frac{1}{f^2} - m_r\right)^{-1/2} > L \quad \text{or} \quad 2d_b > L$$

$$\left(\text{resp. either}\quad \int_{a^*(m_l)}^{\tau_0} f^{-2}\left(\frac{1}{f^2} - m_l\right)^{-1/2} > L \quad \text{or} \quad 2d_a > L\right). \tag{29}$$

As we checked in Lemmas 8 and 9, inequalities (29) allow us to obtain a geodesic joining z_0 and z_0' with component on the fiber a reparameterization of $\hat{\gamma}_F(r)$ (recall that in these lemmas $\hat{\gamma}_F(r)$ was always taken as a minimizing F-geodesic, but the minimizing property was used just to ensure that (29) holds). It is straightforward to check that these inequalities also hold if $b^*(m_r) < b$ or $a < a^*(m_l)$, because the corresponding integral is then infinite.

If (F, g) is complete and F is not contractible then, for fixed $x_0, x_0' \in F$, there exist a sequence of geodesics $\hat{\gamma}_F^m(r)$ joining x_0 and x_0' with diverging lengths L_m (see, e.g. [15, Theorem 2.11.9]). Let $\gamma^m(t) = (\tau^m(t), \gamma_F^m(t))$ be the geodesic connecting z_0, z_0' constructed from $\hat{\gamma}_F^m(r)$, and assume $\tau_0 \leq \tau_0'$. If $\gamma^m(t)$ is causal, then necessarily (7) and (8) hold with $L = L_m$. But in this case $D \leq 0$ and, thus, $L_m \leq \int_{\tau_0}^{\tau_0'} 1/f (< \infty)$. As the sequence $\{L_m\}$ is diverging, all the geodesics but a finite number are spacelike.

The last assertion is also a direct consequence of the fact that the lengths of the F-pregeodesics corresponding to causal geodesics are bounded by $\int_{\tau_0}^{\tau_0'} 1/f$, and Lemma 10. □

Lemma 10. *If (M, g) is a complete Riemannian manifold and $p, q \in M$ are not conjugate, then for all $L > 0$ there exist at most finitely many geodesics with length smaller than L connecting p and q.*

Proof. Otherwise, from the compactness of $\{v \in T_pM : |v| \leq L\}$, we would obtain a sequence $\{v_n\}_{n\in\mathbb{N}}$, $v_n \to v_0$, $v_n, v_0 \in T_pM$ such that $\exp_p(v_n) = \exp_p(v_0) = q$ for all n. Then, v_0 would be a singular point of $\exp_p$ and thus, p and q would be conjugate for the geodesic $\gamma(t) = \exp_p(t \cdot v_0)$, $t \in \mathbb{R}$, which is a contradiction. □

Remark. *In the proof of Theorem 3, we have used that, for a complete Riemannian manifold which is non-contractible in itself, infinitely many geodesics joining p and q exist, and there is a sequence of them with diverging lengths. So, in this case, Lemma 10 says that if p, q are not conjugate then any sequence of geodesics joining them have diverging lengths. In particular, the number of geodesics joining two non-conjugate points of a complete Riemannian manifold must be enumerable.*

5. Conjugate points and Morse-type inequalities

In order to prove results on conjugate points, it seems more natural to consider all the geodesics obtained by varying a fixed one with the same speed D. So, we will drop previous normalization $c = 1$ for geodesics non-tangent to the base. The only modification in previous formulae which we will have to bear in mind is that now (5) reads

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{\sqrt{c}} f^2 \circ \tau \circ t. \tag{30}$$

So the definition of h in (7) must be changed to

$$h^\epsilon = \epsilon \sqrt{c} f^{-2} \left(-D + \frac{c}{f^2} \right)^{-1/2}. \tag{31}$$

Theorem 4. *Let $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0')$ be two points of the GRW space–time $(I \times F, -\mathrm{d}\tau^2 + f^2 g)$ with n-dimensional fiber (F, g). Assume that $\gamma(t) = (\tau(t), \gamma_F(t))$ is a geodesic which joins them, with $\gamma_F(t)$ being the reparameterization of a non-constant F-geodesic $\hat{\gamma}_F$, and that z_0, z_0' are conjugate along γ with multiplicity $m \in \{0, 1, \ldots, n\}$ ($m = 0$ means not conjugate).*

1. *Then x_0, x_0' are conjugate points of multiplicity $m' \in \{m, m-1\}$ along $\hat{\gamma}_F$ (at the corresponding points of the domain). In particular, if z_0, z_0' are non-conjugate then so are x_0 and x_0'.*
2. *If γ is a causal geodesic (or any geodesic without zeroes in $\mathrm{d}\tau/\mathrm{d}t$) then $m' = m$.*

Remark.

1. *The following direct computation shows that even in the excluded case $\hat{\gamma}_F \equiv x_0 = x_0'$ ($\hat{\gamma}_F$ is constant), the points z_0, z_0' are not conjugate. Thus, this case can be included in Theorem 4 with the convention "a constant geodesic $\hat{\gamma}_F$ has no conjugate points". Assume $\tau_0 < \tau_0'$ and consider the geodesic $\gamma(t) = (t, x_0)$, $t \in [\tau_0, \tau_0']$. Let $E_i(t)$, $i \in \{1, \ldots, n\}$ be orthonormal parallel fields along γ which span the orthogonal to γ'. A vector field $J(t) = \sum_i a_i(t) E_i(t)$ along γ is a Jacobi field if and only if each function $a_i(t)$ is a solution of the Sturm differential equation*

$$a''(t) - \frac{f''(t)}{f(t)} a(t) = 0, \quad t \in [\tau_0, \tau_0']. \tag{32}$$

   But, clearly $f(t)$ is also a strictly positive solution of (32). Thus, if $a(\tau_0) = 0$ and $a'(\tau_0) \neq 0$ then $a(\tau)$ cannot vanish on $(\tau_0, \tau_0']$, as required.
2. *Moreover, for any $\tau > \tau_0$, replace (32) by the spectral equation (see [2])*

$$a''(t) - \frac{f''(t)}{f(t)} a(t) + \lambda_\tau\, a(t) = 0, \tag{33}$$

   $\lambda_\tau \in \mathbb{R}$ with boundary conditions $a(\tau_0) = a(\tau) = 0$. A simple Sturm argument shows that if $\tau < \bar{\tau}$ then $\lambda_\tau > \lambda_{\bar{\tau}}$, i.e. the spectral flow $\lambda(\tau) \equiv \lambda_\tau$ is decreasing. This also holds for the static bidimensional case (see Section 6), and should be compared with [2]. At any case, the main result of [2] can be reobtained, as we will see in Section 6, independently, it is also reobtained in [13], in the general setting of geodesics admitting a timelike Jacobi field.

Proof of Theorem 4.

Step 1. For any geodesic γ, $m' \geq m-1$.

Consider $v_0, v_1, \ldots, v_m \in T_{z_0}(I\times F)$ such that $V = \mathrm{Span}\{v_1,\ldots,v_m\}$, where $V = \ker((\mathrm{dexp}_{z_0})_{v_0})$ and $\exp_{z_0}(v_0) = z_0'$. From semi-Riemannian Gauss Lemma [16, Section 5.1], v_0 and each v_i are orthogonal, so $\{v_0,\ldots,v_m\}$ are linearly independent (recall that if v_0 is lightlike then as v_0 and v_i are not collinear then each v_i is spacelike). Moreover, consider the usual projection on the fiber, π_F; as γ is not on the base, then $(\mathrm{d}\pi_F)_{z_0}v_0 \neq 0$ and say $\{(\mathrm{d}\pi_F)_{z_0}v_0, (\mathrm{d}\pi_F)_{z_0}v_1, \ldots, (\mathrm{d}\pi_F)_{z_0}v_{m-1}\}$ are linearly independent. So $(\mathrm{d}\pi_F)_{z_0}v_0$ is parallel to the initial velocity of $\hat{\gamma}_F$, and we have just to prove that there exist a direction of conjugacy of $\hat{\gamma}_F$ between x_0, x_0' in each plane $W_i = \mathrm{Span}\{(\mathrm{d}\pi_F)_{z_0}v_0, (\mathrm{d}\pi_F)_{z_0}v_i\} \subseteq T_{x_0}F$ for $i = 1,\ldots,m-1$.

Defining $\alpha_i(s) = v_0 + sv_i$, we have $\mathrm{d}/\mathrm{d}s|_{s=0}\exp_{z_0}(\alpha_i(s)) = 0$ and, thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \pi_F \circ \exp_{z_0}(\alpha_i(s)) = 0. \tag{34}$$

There exist a non-constant continuous curve $\beta_i(s) \in W_i$, $i = 1,\ldots,m-1$ such that

$$\exp_{x_0}(\beta_i(s)) = \pi_F \circ \exp_{z_0}(\alpha_i(s)). \tag{35}$$

In fact, we take

$$\beta_i(s) = \mu_i(s)\frac{\mathrm{d}\pi_F(\alpha_i(s))}{|\mathrm{d}\pi_F(\alpha_i(s))|}, \tag{36}$$

where $\mu_i(s)$ is the length of the pregeodesic $t \to \pi_F \circ \exp_{z_0}(t\,\alpha_i(s))$ on $[0, 1]$.

Recall that $(\mathrm{d}\pi_F)_{z_0}v_0$ is parallel to $\beta_i(0) \equiv \omega_0$, and we had to prove that $(\mathrm{d}\exp_{x_0})_{w_0}$ restricted to W_i is singular. Otherwise, $\beta_i(s)$ would be smooth around 0 from (35). From (36), $0 \neq \beta_i'(0) \in W_i$, and from (34) and (35), $\beta_i'(0) \in \ker(\mathrm{d}\exp_{x_0})_{\omega_0}$, a contradiction.

Step 2. If γ is causal then $m' \geq m$.

We will check that if γ is not tangent to the base but it is causal (or any geodesic without zeroes in the derivative of the timelike component) then $\{v_1,\ldots,v_m\}$ are tangent to the fiber. So $\{(\mathrm{d}\pi_F)_{z_0}v_0, (\mathrm{d}\pi_F)_{z_0}v_1, \ldots, (\mathrm{d}\pi_F)_{z_0}v_m\}$ are linearly independent and the result follows as in the previous step.

From the hypotheses,

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \epsilon\left(-D + \frac{c}{f^2\circ\tau}\right)^{1/2} \neq 0 \tag{37}$$

for all t, where $D = g^f(\mathrm{d}\gamma/\mathrm{d}t, \mathrm{d}\gamma/\mathrm{d}t)$ and $c = (f^4\circ\tau)g(\mathrm{d}\gamma_F/\mathrm{d}t, \mathrm{d}\gamma_F/\mathrm{d}t)$. Consider the usual projection on the base π_I, we will check that $(\mathrm{d}\pi_I)_{z_0}(v_i) = 0$. Let $\alpha_i(s) \in T_{z_0}(I\times F)$ be a curve such that $\alpha_i(0) = v_0$, $\mathrm{d}/\mathrm{d}s|_{s=0}\,\alpha_i(s) = v_i$, as above, and we also impose $g^f(\alpha_i(s), \alpha_i(s)) = g^f(v_0, v_0)$ for all s. Put $\gamma(s,t) = \exp_{z_0}(t\,\alpha_i(s)) \equiv (\tau_s(t), \gamma_{F_s}(t))$ (thus $D(s) \equiv g^f(v_0, v_0)$). If $\mathrm{d}\pi_I(v_i) = \mathrm{d}/\mathrm{d}s|_{s=0}\,\tau_s'(0) \neq 0$ then, as $D = -\tau_s'(0)^2 + c(s)/f^2(\tau_0)$ is constant, we obtain that $\mathrm{d}/\mathrm{d}s|_{s=0}\,c(s) \neq 0$. Now, including in

(37) the dependence on s we have

$$\int_{\tau_s(0)}^{\tau_s(1)} \frac{\mathrm{d}\tau}{\epsilon(-D + c(s)/f^2(\tau))^{1/2}} = 1,$$

and deriving with respect to s, we obtain $\mathrm{d}/\mathrm{d}s|_{s=0}\, \tau_s(1) \neq 0$. Therefore, $\mathrm{d}/\mathrm{d}s|_{s=0}\, \pi_I \circ \exp_{z_0}(\alpha_i(s)) \neq 0$ which contradicts that v_i is a direction of conjugacy.

Step 3. $m \geq m'$.

Let x_0, x_0' be conjugate points of multiplicity m' along the F-geodesic $\hat{\gamma}_F$ and suppose $\gamma(0) = z_0$, $\gamma(1) = z_0'$. If $\mathrm{Span}\{w_1, \ldots, w_{m'}\} = \ker((\mathrm{d}\exp_{x_0})_{w_0})$ where $\exp_{x_0}(w_0) = x_0'$, consider a curve $\beta_i(s)$ in $T_{x_0}F$ such that $\beta_i(0) = w_0$, $\mathrm{d}/\mathrm{d}s|_{s=0}\, \beta_i(s) = w_i$ and $|\beta_i(s)| = |w_0|$ for all s, $i = 1, \ldots, m'$. Define $\alpha_i(s) \in T_{z_0}(I \times F)$ such that $(\mathrm{d}\pi_I)_{z_0}(\alpha_i(s)) = \mathrm{d}\tau/\mathrm{d}t(0)$ and

$$(\mathrm{d}\pi_F)_{z_0}(\alpha_i(s)) = \frac{\sqrt{c}}{f^2(\tau_0)} \frac{\beta_i(s)}{|w_0|}. \tag{38}$$

For each s, the geodesic on the GRW space–time $\gamma(s,t) = \exp_{z_0}(t\,\alpha_i(s)) \equiv (\tau_s(t), \gamma_{F_s}(t))$ satisfy that $\gamma_{F_s}(t) = \exp_{x_0}(r_s(t)\beta_i(s))$ where $r_s(t)$ is an increasing function, because $\gamma_{F_s}(t)$ is a pregeodesic on the fiber F. But from (4) and (30), $r_s(t)$ is determined just by $c(s) \equiv c$ and $D(s) \equiv D$, so $r_s(t)$ is independent of s, i.e. $r_s(t) \equiv r(t)$. Computing for $s = 0$, it is clear that $r^{-1}(1) = 1$ thus, necessarily $\pi_F \circ \exp_{z_0}(\alpha_i(s)) = \exp_{x_0}(\beta_i(s))$ for all s. As $w_i \in \ker((\mathrm{d}\exp_{x_0})_{w_0})$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \pi_F \circ \exp_{z_0}(\alpha_i(s)) = 0. \tag{39}$$

On the other hand, from the relation between the parameters τ and r for $\gamma(s,t)$ given by (31), we have

$$\int_{\tau_0}^{\tau_0'(s)} \sqrt{c} f^{-2}(\tau) \left(-D + \frac{c}{f^2(\tau)}\right)^{-1/2} \mathrm{d}\tau = |w_0| \quad (= \text{length of } \gamma_{F_s} \text{ for all } s), \tag{40}$$

where the integral is possibly considered under Convention 1. But the integrand and the right-hand side in (40) are independent of s, thus $\tau_0'(s) = \pi_I \circ \exp_{z_0}(\alpha_i(s))$ is constant, and

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \pi_I \circ \exp_{z_0}(\alpha_i(s)) = 0. \tag{41}$$

From (39) and (41) $v_i = \mathrm{d}/\mathrm{d}s|_{s=0}\, \alpha_i(s)$ yields a direction of conjugacy of γ for any $i = 1, \ldots, m'$, and it is clear from the construction that these m' directions are independent. □

Remark. *Note that the following case may hold: the point x_0 has a conjugate point x_1 along the F-geodesic $\hat{\gamma}_F$, but if we consider any geodesic γ emanating from $z_0 = (\tau_0, x_0)$ which projects on $\hat{\gamma}_F$, the reparameterization γ_F of $\hat{\gamma}_F$ does not reach until x_1 and, so, there is no conjugate point z_1 of z_0 along γ which projects onto x_1. That is, the geodesic γ "escapes" at the extremes of I before $\hat{\gamma}_F$ reaches x_1. This possibility may happen, e.g.*

when the space–time is extendible through the extremes of I. But it does not necessarily happen because of this reason; in fact, de Sitter space–time, where $f = \cosh$, *is a simple counterexample* (*recall that if* $\int_c^b f = \infty$ *all null geodesics are future-complete* [18] *and the GRW space–time not only is not extendible through b as a GRW space–time but also it is not extendible as a space–time; compare all this discussion with* [21, p. 73]).*When the fiber is weakly convex, the necessary and sufficient conditions to ensure that, for any geodesic* γ *non-tangent to the base,* γ_F *will cover all* $\hat{\gamma}_F$ *are the "non-escape" equalities*

$$\int_a^c f^{-2}\left(\frac{1}{f^2}+1\right)^{-1/2} = \infty, \qquad \int_c^b f^{-2}\left(\frac{1}{f^2}+1\right)^{-1/2} = \infty \tag{42}$$

for certain $c \in (a, b)$ (*see* [8, *Lemma* 4]). *Recall that this condition implies Condition* (*A*) *and, so the space–time will be geodesically connected*.

Summing up we get the following corollary.

Corollary 1. *Consider a GRW space–time with weakly convex fiber where the "non-escape" equalities* (42) *hold. Then the space–time is geodesically connected and any causal geodesic* $\gamma(t) = (\tau(t), \gamma_F(t))$ *starting at* z_0 *has conjugate points in bijective correspondence* (*including multiplicities*) *with the conjugate points of the inextendible geodesic* $\hat{\gamma}_F(r)$ *obtained from the projection* $\gamma_F(t)$ *on the fiber*.

Remark. *This result allows to extend, in our ambient, the ones by Uhlenbeck* [21] *for null geodesics to all causal geodesics. For instance, normalize all causal geodesics* (*non-tangent to the base*) *such that* $c \equiv (f^4 \circ \tau) g(\mathrm{d}\gamma_F/\mathrm{d}t, \mathrm{d}\gamma_F/\mathrm{d}t) = 1$ *and choose* $D \leq 0$; *all future-pointing causal geodesics starting at* $z_0 = (\tau_0, x_0)$ *and having associated the fixed value of* $D = g^f(\mathrm{d}\gamma/\mathrm{d}t, \mathrm{d}\gamma/\mathrm{d}t)$, *are in bijective correspondence with the F-geodesics starting at* x_0, *being the conjugate points preserved. So*

> *Under the assumptions of Corollary* 1, *and fixed* $D \leq 0$, *if* x_0 *and* x_1 *are not conjugate the loop space of F is homotopic to a cell complex constructed with a cell for each causal D-geodesic* (*with* $c = 1$) *from* z_0 *to the line* $L_{x_1} = \{(t, x_1) : t \in I\}$ *with the dimension of the cell equal to the index of the D-geodesic.*

Recall that in [21], *the conformal invariance of null conjugate points is explicitly used, but this invariance does not hold for timelike geodesics* (*bidimensional anti-de Sitter space–time, which is globally conformal to a strip in Lorentz–Minkowski space–time, is a simple example*); *this makes our approach necessary*.

Theorem 4 and equalities (42) can be also combined to yield Morse relations as follows. Fix two non-conjugate points $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0')$ and a field $\mathcal{K}$. Let $\Omega(z_0, z_0')$ (resp. $\Omega(x_0, x_0')$) be the space of continuous paths joining z_0, z_0' in $I \times F$ (resp. x_0, x_0' in F). Let $\mathcal{P}_{z_0,z_0'}(t)$ (resp. $\mathcal{P}_{x_0,x_0'}(t)$) be the Poincaré polynomial of $\Omega(z_0, z_0')$ (resp. $\Omega(x_0, x_0')$), i.e. $\mathcal{P}_{z_0,z_0'}(t)$ is the formal series

$$\mathcal{P}_{z_0,z_0'}(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots,$$

where β_q is the qth Betti number of $\Omega(z_0, z_0')$ for homology with coefficients in $\mathcal{K}$, $\beta_q = \dim H^q(\Omega(z_0, z_0'), \mathcal{K})$. Clearly, $\mathcal{P}_{z_0,z_0'}(t) \equiv \mathcal{P}_{x_0,x_0'}(t)$. Let

$$\mathcal{M}_{z_0,z_0'}(t) = \bar{a}_0 + \bar{a}_1 t + \bar{a}_2 t^2 + \cdots \qquad (\text{resp.} \quad \mathcal{M}_{x_0,x_0'}(t) = a_0 + a_1 t + a_2 t^2 + \cdots)$$

be the Morse polynomials of z_0, z_0' (resp. x_0, x_0'), i.e. $\bar{a}_q$ (resp. a_q) is the number of geodesics joining z_0 and z_0' (resp. x_0 and x_0') with Morse index equal to q, where the Morse index of a geodesic connecting two fixed non-conjugate points is the sum of the indexes of conjugate points to the first point along the geodesic. Then, under the hypotheses of Theorems 3 and 4, we have

$$a_q \leq \bar{a}_q + \bar{a}_{q+1} \quad \forall q \geq 0, \tag{43}$$

$$\bar{a}_0 > 0 \Rightarrow a_0 > 0, \qquad \bar{a}_q > 0 \Rightarrow a_{q-1} + a_q > 0 \quad \forall q \geq 1. \tag{44}$$

In particular, if the polynomials are finite then $\mathcal{M}_{z_0,z_0'}(t) \geq \mathcal{M}_{x_0,x_0'}(t), \forall t \geq 1$. But if (F, g) is a complete Riemannian manifold, then the well-known Morse relations imply the existence of a formal polynomial with non-negative integer coefficients $\mathcal{Q}(t)$ such that

$$\mathcal{M}_{x_0,x_0'}(t) = \mathcal{P}_{x_0,x_0'}(t) + (1+t)\mathcal{Q}(t). \tag{45}$$

Remark. *In general, it is not true that $a_0 \geq \bar{a}_0$ or $a_{q-1} + a_q \geq \bar{a}_q$. Recall that many geodesics in the GRW space–time connecting z_0, z_0' may project on the same pregeodesic of F. A simple counterexample of this is de Sitter space–time* (*with a straightforward modification, one can also get that hypotheses in Theorem* 3 *are fulfilled*). *So, inequalities* (44) *cannot be improved*.

Summing up we get the following corollary.

Corollary 2. *In a globally hyperbolic GRW space–time satisfying either Condition* (*A*) *or Condition* (*B*) *with d_a, d_b* (*if defined*) *equal to infinity the Morse inequalities* (43) *and* (44) (*with* (45)) *hold*.

As a consequence, if the Morse polynomial $\mathcal{M}_{z_0,z_0'}(t)$ is finite then, for each pair of non-conjugate points z_0, z_0' there exist a polynomial $\mathcal{Q}(t)$ with non-negative integer coefficients and computable from the fiber such that

$$\mathcal{M}_{z_0,z_0'}(t) \geq \mathcal{P}_{z_0,z_0'}(t) + (1+t)\mathcal{Q}(t), \forall t \geq 1.$$

6. Applications

6.1. Two-dimensional case

Next, we will particularize previous results to bidimensional GRW space–times with strongly convex fiber (necessarily an interval $(J, \mathrm{d}x^2)$). Recall that in this case the opposite metric $-g^f$ is also Lorentzian and, in fact, it corresponds to a static (standard) space–time.

The chronological relation can be now extended for non-causally related points, just defining that two points are spacelike related if they are chronologically related for $-g^f$. In fact, we will simplify our terminology with the following (re-)definition.

Definition 5. Consider a bidimensional GRW (or static) space–time. Two points (τ_0, x_0), (τ_0', x_0') are spacelike (resp. timelike, lightlike) related if and only if there exists a spacelike (resp. timelike, lightlike with non-vanishing derivative) curve joining them.

From a direct computation (see also [17, Theorem 3.3 and Lemma 3.5]) we get the following lemma.

Lemma 11. *Given* $(\tau_0, x_0), (\tau_0', x_0') \in (I \times J, -\mathrm{d}\tau^2 + f^2\,\mathrm{d}x^2)$, *they are*
1. *spacelike related if and only if* $\int_{\tau_0}^{\tau_0'} f^{-1} < d(x_0, x_0')$;
2. *lightlike related if and only if* $\int_{\tau_0}^{\tau_0'} f^{-1} = d(x_0, x_0')$;
3. *timelike related if and only if* $\int_{\tau_0}^{\tau_0'} f^{-1} > d(x_0, x_0')$.

Now, as a consequence of Lemma 11 and Theorem 2 we get the following corollary.

Corollary 3. *In a GRW space–time* $(I \times J, -\mathrm{d}\tau^2 + f^2\,\mathrm{d}x^2)$
1. *If* $(\tau_0, x_0), (\tau_0', x_0')$ *are timelike (resp. lightlike) related then there exist a unique geodesic (necessarily timelike (resp. lightlike)) which joins them.*
2. *All* $(\tau_0, x_0), (\tau_0', x_0')$ *which are spacelike related can be joined by a geodesic (necessarily spacelike) if and only if Condition (C) or Condition (R) holds.*

From Theorem 4 and the fact that there are no conjugate points on a manifold of dimension 1, we get the following corollary.

Corollary 4. *In a GRW space–time* $(I \times J, -\mathrm{d}\tau^2 + f^2\,\mathrm{d}x^2)$ *no geodesic* $\gamma(t) = (\tau(t), \gamma_F(t))$ *without zeroes in* $\mathrm{d}\tau/\mathrm{d}t$ *have conjugate points.*

In particular, causal geodesics are free of conjugate points.

Now, consider a bidimensional static space–time, say $(K \times J \subseteq \mathbb{R}^2, g_S = \mathrm{d}y^2 - f^2(y)\,\mathrm{d}x^2)$, where g_S can be seen as the reversed metric of a GRW space–time. Summarizing the conclusions of Lemma 11 and Corollaries 3 and 4, the following extension of Theorem 1.1 in [5] can be given (see also [13, Proposition 6.6]).

Corollary 5. *Given* $(y_0, x_0), (y_0', x_0')$ *in the static space–time* $(K \times J \subseteq \mathbb{R}^2, g_S = \mathrm{d}y^2 - f^2(y)\,\mathrm{d}x^2)$, *they are*
1. *Spacelike related if and only if* $\int_{y_0}^{y_0'} f^{-1} > d(x_0, x_0')$. *In this case there exist a unique geodesic which joins them; this geodesic is necessarily spacelike and without conjugate points.*
2. *Lightlike related if and only if* $\int_{y_0}^{y_0'} f^{-1} = d(x_0, x_0')$. *In this case there exist a unique geodesic which joins them; this geodesic is necessarily lightlike and without conjugate points.*

3. *Timelike related if and only if* $\int_{y_0}^{y_0'} f^{-1} < d(x_0, x_0')$. *All points which are timelike related can be joined by a geodesic* (*necessarily timelike*) *if and only if Condition* (*C*) *or Condition* (*R*) *holds*.

Remark. *In fact, no geodesic of the static space–time without zeroes in the derivative of its spacelike component has conjugate points. Anti-de-Sitter space–time is an example of static space–time where all the timelike geodesics have conjugate points. Moreover, it is not geodesically connected*.

6.2. Conditions on curvature

As commented in Section 1, it is natural to assume, for a realistic GRW space–time that $\mathrm{Ric}(\partial_t, \partial_t) \geq 0$, and it is straightforward to check that this condition is equivalent to $f'' \leq 0$ (see [16, Corollary 7.43]). Recall that in this case $\lim_{\tau\to a,b} f'$ and $\lim_{\tau\to a,b} f$ always exist. So taking into account the cases in Table 1 we see that Condition (A) always holds except when $b < \infty$ (resp. $a > -\infty$) and $f'(b) > 0$ (resp. $f'(a) < 0$). In this case, although the GRW space–time is not geodesically connected, it is possible to extend the warping function f through b (resp. a) obtaining thus an extended space–time, which is also GRW. The GRW space–time will be called *inextendible* if whenever an extreme of I is finite, then f cannot be extended continuously at these extremes to a real value $\alpha > 0$. It seems clear that from a physical viewpoint just inextendible GRW space–times must be taken into account.

Therefore, Theorems 1 and 3 are applicable to these inextendible GRW space–times, yielding the points (1) and (4) in Corollary 6 (the other two are included for the sake of completeness).

Corollary 6. *An inextendible GRW space–time with* $\mathrm{Ric}(\partial_t, \partial_t) \geq 0$ *and weakly convex fiber satisfies the following*:

1. *Each two causally related points can be joined with one non-spacelike geodesic, which is unique if the fiber is strongly convex*.
2. *The space–time is geodesically connected. Moreover, each strip* $(\hat{a}, \hat{b}) \times F \subset I \times F$, $a < \hat{a} < \hat{b} < b$ *with the restricted metric is geodesically connected if and only if* $f'(\hat{a}) \geq 0$ *and* $f'(\hat{b}) \leq 0$ (*i.e.* $f'(\hat{a}) \cdot f'(\hat{b}) \leq 0$).
3. *There exist a natural surjective map between geodesics connecting* $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0') \in I \times F$ *and F-geodesics connecting* x_0 *and* x_0'. *Under this map, when the geodesic connecting* z_0 *and* z_0' *is causal then the multiplicity of its conjugate points is equal to the multiplicity for the corresponding geodesic connecting* x_0, x_0'.
4. *If* (F, g) *is complete and F is not contractible in itself, then any* $z_0, z_0' \in I \times F$ *can be joined by means of infinitely many spacelike geodesics. If* x_0, x_0' *are not conjugate there are at most finitely many causal geodesics connecting them*.

For the last assertion (2), recall that it is straightforward from Theorem 2 under strongly convexity. But, from the proof of this theorem, this assumption can be dropped because

$f'' \leq 0$ (recall that then Conditions (A), (B), (C) are equivalent and Condition (R) is not applicable).

Finally, we give a further consequence of equalities (42).

Corollary 7. *Consider a GRW space–time* $(I \times F, g^f)$ *which is globally hyperbolic and satisfies the non-escape equalities* (42), *and fix* $D_0 \leq 0$. *If any geodesic* $\gamma(t) = (\tau(t), \gamma_F(t))$ *starting at* $z_0 = (t_0, x_0)$ *and having associated values D, c equal to* $D_0, 1$, *respectively, is free of conjugate points then the fiber can be covered topologically by* $\mathbb{R}^n$, *being* $n = \dim F$.

Proof. Under this assumption the F-geodesics starting at x_0 have no conjugate points and so, as F is complete, $\exp_{x_0} : T_{x_0}F \equiv \mathbb{R}^n \to F$ is a surjective local diffeomorphism. Taking the pull-back metric on $T_{x_0}F$, a local isometry with a domain complete manifold (and so a Riemannian covering) is obtained. □

Remark. *The assumption on conjugate points when* $D_0 = 0$ *holds if in the future of* z_0 *we have* $R(X, Y, Y, X) \leq 0$ *whenever X, Y span a degenerate plane on a lightlike geodesic starting at* z_0 (*see* [3, *Theorem* 10.77]); *moreover, the non-escape inequalities* (42) *can be reduced to*

$$\int_a^c f^{-1} = \infty, \qquad \int_c^b f^{-1} = \infty, \tag{46}$$

when just null geodesics are considered, so we reobtain [21, *Theorem* 5.3] *in our ambient*.

Acknowledgements

This research was partially supported by an MEC Grant No. PB97-0784-C03-01.

References

[1] L.J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson–Walker space–times, Gen. Relativity Gravitation 27 (1995) 71–84.
[2] R. Bartolo, A. Germinario, M. Sánchez, Convexity of domains of Riemannian manifolds, Preprint, 1999.
[3] J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, Monographs Textbooks in Pure and Applied Mathematics, Vol. 202, Dekker, New York, 1996.
[4] V. Benci, D. Fortunato, Existence of geodesics for the Lorentz metric of a stationary gravitational field, Ann. Inst. Henri Poincaré 7 (1990) 27–35.
[5] V. Benci, F. Giannoni, A. Masiello, Some properties of the spectral flow in semi-Riemannian geometry, J. Geom. Phys. 27 (1998) 267–280.
[6] V. Benci, A. Masiello, A Morse index for geodesics in static Lorentzian manifolds, Math. Ann. 293 (1992) 433–442.
[7] E. Calabi, L. Markus, Relativistic space forms, Ann. Math. 75 (1962) 63–76.
[8] J.L. Flores, M. Sánchez, Geodesic connectedness of multiwarped space–times, Preprint, 1999.
[9] F. Giannoni, Geodesics on non-static Lorentz manifolds of Reissner–Nordström type, Math. Ann. 291 (1991) 383–401.
[10] F. Giannoni, A. Masiello, Geodesics on Lorentzian manifolds with quasi-convex boundary, Manuscripta Math. 78 (1993) 381–396.

[11] F. Giannoni, A. Masiello, P. Piccione, Convexity and the finiteness of the number of geodesics: applications to the multiple-image effect, Class. Quantum Gravity 16 (1999) 731–748.
[12] F. Giannoni, A. Masiello, P. Piccione, A Morse theory for massive particle and photons in general relativity, J. Geom. Phys., in press.
[13] F. Giannoni, A. Masiello, P. Piccione, D. Tausk, A generalized index theorem for Morse–Sturm systems and applications to semi-Riemannian geometry, Preprint, 1999.
[14] A.D. Helfer, Conjugate points on spacelike geodesics or pseudo-self-adjoint Morse–Sturm–Liouville systems, Pacific J. Math. 164 (1994) 321–350.
[15] A. Masiello, Variational Methods in Lorentzian Geometry, Pitman Research Notes in Mathematics Series, Vol. 309, Longman, Harlow, Essex, UK, 1994.
[16] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Series in Pure and Applied Mathematics, Vol. 103, Academic Press, New York, 1983.
[17] M. Sánchez, Geodesic connectedness in generalized Reissner–Nordström type Lorentz manifolds, Gen. Relativity Gravitation 29 (1997) 1023–1037.
[18] M. Sánchez, On the geometry of generalized Robertson–Walker space–times: geodesics, Gen. Relativity Gravitation 30 (1998) 915–932.
[19] M. Sánchez, On the geometry of generalized Robertson–Walker space–times: curvature and Killing fields, J. Geom. Phys. 31 (1999) 1–15.
[20] H.-J. Schmidt, How should we measure spatial distances? Gen. Relativity Gravitation 28 (7) (1996) 899–903.
[21] K. Uhlenbeck, A Morse theory for geodesics on Lorentz manifolds, Topology 14 (1975) 69–90.